Newcastle 1st Dist.
Durham Colln.

No 1815 Receipt

Excise-Office, at Newcastle upon Tyne
this 28 Day of July 1756

Received of Mrs Hannah Dagniaa — — —
of Newcastle upon Tyne in the Parish of All
Saints — the Sum of Five Shillings —
for One Hundred — — — — / — —
Ounces of Silver Plate, of which she hath this Day given notice, according to the Statute of the Twenty-Ninth Year of His present Majesty. In full for one Year, ending on the 5th Day of July — next ensuing.

Tho: Wellerton

Thos Brewster

[illegible]

South Sheeles. The Second — day of June . . 1760.	l.	s.	d.
RECeived of Mrs Dagnia . . . . the Sum of Seven pounds three Shill. & eightpence ½ for One Year's Rent, and By Rent due to the Honourable the DEAN and CHAPTER of Durham, at Pentecost last, for a Tenement in a Tenement & Allymonde at Harton and a Tenement at South Sheeles.	7 :	3	8½
Acquitt.	"	3 :	2
	7 :	6 :	10½

Received by Tho: Hogg

YOU are hereby Required to pay this Rent, due at Martinmas and Pentecost . . next, at Mrs Readheads in South Sheeles on the 18 Day of May — 1761 otherwise the same will be Distrain'd for, without further Notice.

THE
THEORY and PRACTICE
OF
GAUGING,

Demonstrated in a

SHORT and EASY METHOD.

CONTAINING, among other PARTICULARS;

The Method of Computing Decimally; extracting the SQUARE ROOT, with the Construction and Use of the SLIDING-RULE explain'd at large.

The Elementary Properties of the CONIC SECTIONS, and the Manner of describing them in Plano.

General Principles of MENSURATION, with Theorems for Measuring all right-lin'd Planes, Parallelopipedons, Prismatic Solids, the Conic Sections, Conoids, Spindles, their Segments, and Frustums, *&c.*

A General Proposition for measuring the Hoofs or Ungula's of any CONE or PYRAMID; and a new Method for Measuring by Approximation.

With the APPLICATION of the preceeding Principles to Practice; and illustrated throughout by particular Examples.

Published with the particular Approbation of the Honourable COMMISSIONERS OF EXCISE.

Design'd for the Use of the Officers of that Revenue.

By *ROBERT SHIRTCLIFFE.*

LONDON:

Printed by H. WOODFALL, without *Temple-Bar*, for the AUTHOR: And are to be had at his House in *Portland-street*, the Corner of *Mortimer-street*, near *Oxford-Market*.

M DCC XL.

To the HONOURABLE

CHARLES POLHILL *Esq*;
JOHN FOWLE *Esq*;
JAMES VERNON *Esq*;
ROBERT EYRE *Esq*;
HORATIO TOWNSHEND *Esq*;
Sir THOMAS ROBINSON *Bart.*
WILLIAM BURTON *Esq*;
JOHN ORLEBAR *Esq*;

COMMISSIONERS

AND

GOVERNOURS

Of and for the MANAGEMENT and RECEIPT of His MAJESTY's Revenues of the Excise, Malt, *&c.*

Within that Part of *Great-Britain* call'd *ENGLAND*;

The following Treatise is, with the greatest Submission, humbly inscrib'd by

Their most Obedient,

and most Faithful Servant,

ROB. SHIRTCLIFFE.

History of science.
Miles
11-2-23
9E53

PREFACE.

IN the following Treatise are laid down the Principles for the Menſuration of Magnitude, with ſuch other Parts of the Mathematicks only, as are neceſſarily leading thereto, and their Application in the Inveſtigating and Demonſtrating the ſeveral Theorems for meaſuring thoſe plain Surfaces and Solids, which more immediately belong to Gauging: The whole being originally deſign'd for the benefit of the Officers of the Exciſe, that they might not only be acquainted with the moſt eaſy and exact Methods for Practice, but be at the ſame time certain of the Truth of the Rules made uſe of.

Care has been taken to avoid meddling with the Methods practiſed by ſome others; or any way to be concerned in refuting their Errors; but here the only buſineſs is, to give the neceſſary Rules for the juſt and ready Practice of the Art of Gauging; ſhewing the Manner of coming at thoſe Rules, from Principles within themſelves the moſt perfect of all human Knowledge.

For that End, it is hoped, every thing here treated of, is explain'd in ſo eaſy a manner, as to

 be

be within the reach even of Beginners, that will give that Attention which is unavoidably required in the Study of every Part of the Mathematicks: And it may with safety be affirm'd, that without a competent Skill in Algebra and Geometry, it is absolutely impossible for any Person to determine, whether the Rules given by common Writers upon this Subject be true or false, and much less to make any (even the least) Improvement in this Part of Science.

Tho' the Practice of Gauging may be allowed to be a sufficient Qualification for every Officer: Yet, 'tis well known from common Experience, that, by far the greatest Part of them, neither want Abilities to understand, or Ambition to have a reasonable Proof of the Truth of the Rules and Methods on which the Practice is founded.

And as no Writer hitherto has thought fit to give both the Theory *and* Practice of Gauging *in one and the same Book, I hope that may be look'd on as a sufficient Apology for my Undertaking to do what I here offer, which is divided into three Parts.*

The first, as an Introduction to the other Parts, containing the Method of computing decimally, by the Pen, with all the useful Contractions: And by the Sliding-Rule with its Constructions, and a very full and particular Account of its Use, together with the Reason of the several Operations made by each Method.

In

In the second Part, there are some of the primary Principles of plain Geometry, with those of the Conic Sections, as preparatory to what follows: Where such general Propositions, for the Measuring of Plane Surfaces and Solids, are laid down, that the Rules for most of the common ones, are deduced by way of Corollaries.

You have here particular Rules for Mensuration of all right-lined Plane Surfaces, Circles, Conic Sections, and for Solids, such as Parallelopipedons, Prisms, Conoids, Spindles, with the Segments and Frustums of these: Some of these perhaps were not published before, such I suppose, are the Series for the Circle (in Pag. 100.) *for the Hyperbola (in Pag.* 109.) *and the Theorems for measuring the Circular, Elliptic, and Hyperbolic Spindles (in Pag.* 156.) *Then are given the Investigation of Theorems, for reducing these Solids nearly into Cylinders of the same Length: From whence is deduced, occasionally, the common fixed Multipliers, and shewn how far they may be depended upon, in the Practice of Gauging. Next, a general Theorem is given, for the Mensuration of all Pyramidical or Conic Hoofs, and the same is accomplish'd in the three Hoofs of an upright Cone, and that of the Square Pyramid: On this depends the method of Gauging an inclined Pyramidical Conical Tun.*

Then the Method is shewn of Measuring by Approximation, which, if brought into Use,

A 2 *would*

would render the Art of Gauging far more compleat, than it has been hitherto: For by this Method, the Measure of any Cask, Tun, Copper, or Still may be found within any desired Degree of Truth, and that by only taking a sufficient Number of Dimensions, which will ever be less than what is required in the common methods of gauging Coppers, and the Conclusions will be much nearer the Truth. But for Casks, there is, besides the usual Data, *required only a Diameter in the Middle between the Head and the Bung; from which the Form of the Cask, in some Degree, is determined.*

This perhaps may at first be look'd upon as a Design to introduce Novelty in a Matter sufficiently confirm'd by Experience, and by that means be rejected as a speculative Nicety.

But if it be considered that the present Practice of Gauging depends upon assigning some known Form to the Tun or Cask to be gauged; and without assuming that Form, we cannot make one Step towards computing its Measure: So that if we are not exact in our Assumption, it is not possible we should be exact in its Content. And it may be with truth affirm'd, that there never was, strictly speaking, a Cask in the Shape of any one of those Solid Figures whose Names they bear; since it never could happen so, unless perfectly by Accident: The Maker having no thought of designing them as such; and besides, should a Cask approach

nearly

nearly in Figure to such a Solid, yet the Officer has no manner of Rule to assist him in ascertaining the same; so that it is but mere guessing at best.

But by the Method I here propose, there is always a Certainty, either to have the Content exactly true, or exceedingly near the Truth: neither is there any Difficulty in taking the fourth Dimension required among the Data; *and the Operation will be very easy by the Sliding-Ruler.*

The third Part of this Book is wholly taken up with the Practice of Gauging; and is only a more full and particular Illustration of what was delivered in the Second Part: The Measure here being estimated by Gallons or Bushels, which, there, were Cubical Inches. Here is shewn the Manner of Cask-Gauging; First, on the Supposition of a Cask having a known Form: Secondly, Without any Regard to the Form, by the Help of a Fourth Dimension taken; so that every one may follow that which seems to him best.

Then the Practice of Ullaging Casks standing or lying is delivered in a Method, which will be found more universal and exact, than what is given for that Purpose, by the Line mark'd Seg. sta. *or* Seg. ly. (*which signifies* Segments standing, *or* Segments lying) *on the Sliding-Ruler; which certainly can serve but one sort of Casks, and that must be similar to the Cask*

Cask from whence the Lines themselves were made.

Afterwards follows the several Rules and Precepts for Gauging Tuns, Coppers, Stills, Cisterns, &c. *with Examples at large to each. And that nothing might be wanting on my Part to render this Treatise as compleat as possibly I could, I have carefully consulted all that has been wrote hitherto upon the Subject, and particularly* J. Kepler, P. Guildin, J. Wallis, W. Jones, —— Sharp, *and* J. Mat. Hasius; *being the most considerable Authors who have delivered any thing to the purpose about this Affair, and at the same time have demonstrated the Rules they gave.*

From these I have selected whatever was thought might conduce either to the Improvement of the Practice of Gauging, or that might make the Reason of the Methods already given more easy to be comprehended.

And as to the Way taken in finding out the several Rules herein delivered, I have not scrupled to make use of either the Arithmetic of Infinites, or the Method of Increments, according as I imagined the one or the other would render the Investigation the most easy. In this I have followed the Example of one of the ingenious Authors above-mentioned, whose Judgment in these Matters will never be doubted, he having deduced the Solutions of several Pro-

Propositions in his Synopsis *after the same manner.*

I hope the Reader will favourably pass over any Inaccuracy of Style he may possibly meet with, and excuse such Errors as too often occur in Subjects of this Nature.

October 30, 1740.
Portland-street, Corner of *Mortimer-street*, near *Oxford-market*.

THE

* The Theorem, *p.* 272. is deduced from *Cor.* 2. *Page* 191. thus, let MN*nm* (*Fig.* 74.) be any Part of the Frustum of a Sphere, and $m'n'$ a Diameter thereof, in the Middle betwixt the extreme ones MN,*mn*, continue *nm*, and thro' m' draw a Line perpendicular to *mn*,MN, let it meet the first in R, the other in *q*, and the Circumference in P; also put $mn=y$, $MN=b$, $Oo=l$, $Rq=l$, $qP=v$, and $m'n'=m$ the Mid. Diam. then by *Corol.* 2. *Page* 191. the Measure of $MmnN=\overline{y^2+4m^2+b^2}\times\frac{al}{6}$. Now 'tis manifest from 3. *Eucl.* 36. $Rn\times Rm=RP\times m'P$; also by the 35 of the same *Elements*, $m'q\times qP=Mq\times qN$; but $Rn=y+\frac{m-y}{2}=\frac{m+y}{2}$, $Rm=\frac{m-y}{2}$, $RP=l+o$; $Rm'=\frac{l}{2}$, also $m'q=\frac{l}{2}$, $qp=v$, $Mq=\frac{b-m}{2}$, $qN=m+\frac{b-m}{2}=\frac{b+m}{2}$, hence the above Equations give $\frac{m+y}{2}\times\frac{m-y}{2}=\frac{m^2-y^2}{4}=\overline{l+v}\times\frac{l}{2}$ and $\frac{b^2-m^2}{4}=v\times\frac{l}{2}$ the latter taken from the former leaves $\frac{2m^2-b^2-y^2}{4}=\frac{l^2}{2}$, therefore $4m^2=4l^2+2b^2+y^2$, put this for $4m^2$ in the above Expression, then we have the Measure of $MmnN=\overline{b^2+y^2+4l^2+2b^2+2y^2}\times\frac{al}{6}=\overline{\frac{b^2+y^2}{2}+\frac{2l^2}{3}}\times al$ in cubic Inches, or $\overline{\frac{b^2+y^2}{2}\ \frac{2l^2}{3}}\times l\times{,}0034$. *Q. E. O.*

THose that are inclined to have the Sliding-Rule, as constructed *Page* 240. may have it accurately made by the ingenious Mathematical Instrument-makers, Mr. *John Coggs* and Mr. *William Wyeth*, near St. *Dunstan*'s Church in *Fleetstreet*,

And by Thos. Cooke in the Old Jury

ERRATA.

PAge 3. *line* 24, *for* 4, three, *read* 3. four. *P.* 14. *l.* 28. *r.* $\frac{p}{10}+\frac{s}{100}$. *p.* 14. *l.* 29. *for* $\frac{S}{5}$, *r.* $\frac{s}{5}$. *p.* 15. *l.* 5, 6, 7, 8, 9, 10. *for* $r+c$, *r.* $r+t$. *l.* 6. *r.* $\frac{10r+t}{25}$. *p.* 19. *l.* 28. *for* right, *r.* left. *p.* 34. *l.* 31. for $\frac{3}{40}$, $\frac{4}{40}$, *r.* $\frac{3}{20}$, $\frac{4}{20}$. *p.* 35. *l.* 26. *for* on, *r.* to. *p.* 37. *l.* 18. *for* 1 6. *r.* 36. *l.* 31. *r.* first, 1. on. *p.* 39. *l.* 9. *r.* represented by a. *l.* 18. *r.* distance. *p.* 42. *l.* 5. *r.* $\frac{nb}{ma}\times\frac{m}{n}\times\frac{b}{a}=$. *p.* 45. *l.* 19. for C, r. *c.* *for* $-2L:c$. *r.* $2L:a+L:c$. *p.* 49. *l.* 8. *for* —, *r.* $=$. *p.* 50. *l.* 17. *for* proportionals, *r.* Propositions. *p.* 51. *l.* 29. for *b*, r. *l.* *p.* 52. *l.* 4. *for* L, *r.* *l.* *p.* 53. *l.* 26. *dele* each of. *p.* 55. *l.* 8. for their Division, *r.* Divisors. *p.* 57. *l.* 31. *r.* surface. *p.* 64. *l.* 19. *for* B*a* 'tis, *r.* B and *dc* drawn 'tis, *p.* 66. *l.* 5, for AC, *r.* AP. *p.* 67. refer to Fig. 17, 18, *l.* 34. *r.* the half Sum. *p.* 71. *l.* 26. *r.* Fig. 21. *l. ult.* *r.* right Lines. *p.* 83. *l.* 19. *r.* an Hyperbola. *p.* 86. *l.* 6. for *v*, r. *s.* *p.* 87. *l.* 17. for r^{m-2}, r. r^{m-5}. *p.* 88. *l.* 2. for *r.* r. *n.* *l.* 16. *for* of, *r.* a. *p.* 92. *l.* 4. *for* $1\times1\times1$, *r.* $1+1+1$. *p.* 90. *l.* 20. *dele* are. *p.* 109. *l.* 10. for mv, r. mv^m. *p.* 110. *l.* 18. *for* foll. *r.* above. *p.* 115. *l.* 3. *for* square, *r.* solid. *p.* 121. *l.* 33. *for* and, *r.* add. *p.* 125. *l. antepenult.* *for* frustum, *r.* conoid. *p.* 126. *l.* 6. *for* conoid, *r.* frustum. *p.* 128. *l.* 13. *for* 216. *r.* 816. *p.* 136. *l.* 18. *for* 2.3. *r.* 1. 3. *p.* 141. *l.* 14. *for* rb^2, *r.* $3b^2$. *l.* 17. *for* $z\times\sqrt{oz-4z^2}$, *r.* $y\sqrt{b^2-4z^2}$. *p.* 142. *l.* 12. for $\frac{b}{r2}$, r. $\frac{b}{12}$. *p.* 144. *l.* 5. for *p.* r. *q.* *p.* 152. *l.* 7. *for* respectively, *r.* on *c* respectively. *p.* 156. *l.* 10. *for* —, *r.* $+$. *p.* 159. *l.* 7. for $y=24$, r. $b=24$. *p.* 165. *l.* 2. for $6m^d$, r $6m$. *l.* 3. for $\frac{14}{13}\times\frac{0}{138}$, r. $\frac{2}{13}\times\frac{60}{40}$. *p.* 173. *l.* 19. *after* isosceles, *r.* and the base circular. *p.* 176. *l.* 9. *for* D, *r.* d. *p.* 183. *l.* 6. *for* Hoof, *r.* Frustum Hoof. *p.* 192. *l.* 13. *for* last, *r.* IV. *l.* 26. *r.* Axis of the Section. *p.* 193. *l.* 7. *for* WB, *r.* W= B. *l.* 8. *for* Line, *r.* Cone. *p.* 194. *l.* 25. *for* much, *r.* little. *p.* 199. *l.* 27. *for* 58. *r.* 59.

THE
THEORY and PRACTICE
OF
GAUGING.

PART I.

CHAP. I.

Of Decimal Fractions.

DECIMAL Fractions are of ſuch great Importance not only in Gauging, but all kind of numeral Computations, where Fractions are concerned, the Operations being performed with the ſame eaſe as by whole Numbers only; and the Sliding-Rule, on which the Practice of Gauging ſo much depends, being decimally divided, it ſeem'd unavoidable in this Treatiſe to give a ſhort Account of *Decimal Arithmetick*. But ſince in the common Occurrencies in life, the Integer is ſeldom decimally divided, for a *Pound Sterling* is divided into 20 equal Parts,

each of which is one Shilling; a Shilling into 12 equal Parts, each of which is one Penny; and each of these again into four Parts, each of which is one Farthing: So a Foot is divided into 12 equal Parts, each of which is one Inch; a Year into 365, each of which is one Day, and so on. We must therefore in the first place say something of dividing an Integer in general; for every Whole may be conceived as divisible into any Number of equal Parts, any Multitude of which, less than the Whole, is in general (properly) call'd a *Fraction*. But it is manifest in order to form an Idea of the Magnitude or Value of any given Number of Equal Parts of any Whole, we must not only know into how many Parts it is divided, but also how many of them we would express; thence it is, that to denote any Fraction, we are obliged to use two Numbers, each of which has its peculiar office, the one to signify into *how many* equal Parts the whole is divided, called the *Denominator*; and the other the *Number of them* we want to express by the Fraction, called the *Numerator*; the latter of which is ever placed above the former.

Ex. gr. If the Integer was divided into four Parts, then one of them is expressed thus $\frac{1}{4}$, two two of them thus $\frac{2}{4}=\frac{1}{2}$, and three of them thus $\frac{3}{4}$, of which 1, 2, 3 are the Numerator, and 4 the Denominator.

But if the Integer be supposed divided decimally, that is into 10, 100, 1000, &c. equal Parts, then any Number of these, less than the whole, is called a *Decimal Fraction*: In this way of dividing the Integer since the Denominator is ever 1, with a certain Number of Cyphers, therefore here to know the Value of any Fraction, we need only know

know the Numerator, and the Number of Cyphers in the Denominator, and that may be done by prefixing Cyphers to the Numerator, till they with the Numerator, be equal in Number to the Cyphers in the Denominator ; thus, $\frac{5}{10}$ is expreſs'd by ,5 ; $\frac{5}{100}$ by ,05 ; and $\frac{5}{1000}$ by ,005 ; ſo on the contrary, from any Decimal Fraction given, we may go back and reduce it to the Form of a Vulgar one ; thus, 15 is equal to $\frac{15}{100}$: 05$=\frac{5}{100}$; ,012$=\frac{12}{1000}$, *&c.* This Thought (to whomſoever we are indebted for it) is no more than a Continuation of the common Scale, which is alſo Decimal ; for in that any Figure is but $\frac{1}{10}$th, $\frac{1}{100}$th, $\frac{1}{1000}$th, what it would be if placed one, or two, or three, *&c.* places more towards the left Hand ; and thus by going on till we paſs the Place of Units, we then fall in with the Fractional Scale.

Thus in the Expreſſion 6543210,123456 ; to expreſs the Value of any particular Figure of the integral Part, we *annex* to the Figure itſelf as many Cyphers as it is diſtant from (0,) the Place of Units, ſo to expreſs the fourth Figure from 0, which here is 4, we muſt *annex* three Cyphers, and it becomes 4000 ; but if we would expreſs the fourth Figure in the Decimal Part from 0, we muſt *prefix* three Cyphers, and it becomes ,0004 ; ſo that the common Scale continued below Units is as here under.

	Millions.	*Hundred Thouſands.*	*Ten Thouſands.*	*Thouſands.*	*Hundreds.*	*Tens.*	*Units.*	*Tenths.*	*Hundreds.*	*Thouſands.*	*Tens of Thouſands.*	*Hundred Thouſant.*	*Millions.*	
&c.	6	5	4	3	2	1	0,	1	2	3	4	5	6	*&c.*

 From

From what has been ſaid it appears that Fractions commonly firſt preſent themſelves in the common Form, therefore in the following Propoſition we ſhall ſhew how to reduce them into Decimals.

PROP. I.

Any Vulgar Fraction being given, to reduce it to a Decimal Rule.

Annex to the Numerator a competent Number of Cyphers, let this be a Dividend, and the Denominator the Diviſor; the Quotient of this Diviſion if there is no Remainder, is the Decimal ſought: But ſhould there be a Remainder, you may approach to any degree of Exactneſs, by annexing more Cyphers.

Examples.

Let it be required to reduce $\frac{3}{4}$, $\frac{5}{8}$, and $\frac{9}{16}$ to Decimals.

Operation.

```
4) ,300 (,75=3/4        8) 5000 (,625=5/8
    28                     48
   ----                   ----
     20                    20
     20                    16
   ----                   ----
      0                     40
                            40
                           ----
                             0
```

$$
\begin{array}{l}
16)\ 900000\ (,5625=\tfrac{9}{16} \\
\quad\ \ 80 \\
\hline
\quad\ \ 100 \\
\quad\ \ \ \ 96 \\
\hline
\quad\ \ \ \ \ 40 \\
\quad\ \ \ \ \ 32 \\
\hline
\quad\ \ \ \ \ \ 80 \\
\quad\ \ \ \ \ \ 80 \\
\hline
\quad\ \ \ \ \ \ \ 0
\end{array}
$$

The Invention of this Rule is as follows. Let $\frac{n}{d}$ be any Vulgar Fraction which is to be reduced to a Decimal, and put x for the Numerator or Decimal sought, then 'tis plain $\frac{n}{d}=\frac{x}{1000}$, *&c.* and multiplying the Equation by 1000, *&c.* we have $x=\frac{n\times 1000}{d}$, *&c.* but to multiply any Number by 1000, *&c.* is only to annex Cyphers to the Number itself, whence the Rule is manifest.

Having now shewn how to turn any vulgar Fraction into a Decimal, in the next place we must shew the Method of computing by them after they are so reduced.

PROP. II.

Any Number of Decimal Fractions being given, it is required to find their Sum?

Set them under one another, as in common Arithmetick, *viz.* tenths under tenths, hundreds under hundreds, thousands under thousands, *&c.* then add them together as whole Numbers, and

place the decimal Point of the Sum directly under thoſe of the Terms, ſo ſhall you have the Sum required.

Examp. I.	II.	III.	IV.
,1543	1,4238	64,521	743,
,2754	,5784	87,043	82,47765
,0132	7,2189	7,218	1549,32187
,8054	6,5432	054	8,0431
,7268	,0741	65,—	,07285
1,9751	15,8384	223,836	2382,91432

Subſtraction of Decimals.

PROP. III.

Any two decimal Fractions being given, to find their Difference; or two Quantities being given, compoſed of Integers and Decimals, from the greater to ſubſtract the Leſſer.

Rule. Place the Terms under one another by the ſame Rule that was given in the above Propoſition, then ſubſtract them as if Integers, placing the decimal Point directly under thoſe above, ſo ſhall you have the Difference ſought.

	Examp. I.	II.	III.
From	,5743	8,7284	460,0954
Subſtract	,2579	1,5896	29.7387
Remainder	,3164	7,1388	430,3567

IV.

9438,0000
72,5847
9365,4153

Mul-

Multiplication of Decimals.

PROP. IV.

Two Quantities composed of Integers and Decimals, or Decimals only, being given, to find their Product.

Set the Units Place of the Multiplier under the laſt Figure of the Multiplicand, and the reſt in order; then multiply as in common Arithmetick, always placing the firſt Figure of every Multiplication under the multiplying Figure, ſo ſhall each Figure of theſe Products when added together, be of the ſame Name with that Figure of the Multiplicand, under which they ſtand.

Examp. I.	II.	III.
1,305	,3754	,135272
56,3	: : :0,754	: : : : :0,00425
3915	: : 15016	: : : : : 676360
7830 :	: 18770 :	: : : : 270544 :
6525 : :	26278 : :	: : : 541088 : :
73,4715	,2830516	,00057490600

Corol. Whence the Number of decimal Places in the Product, is equal to the Sum of decimal Places in the Factors.

When the Multiplicand and Multiplier together contain more Decimals than we would chuſe to have in the Product, they may be contracted to any deſired Number, and yet the Product be as true to ſo many Figures, as if the Factors had been multiplied at large; for which obſerve this *Rule.* Under that Place of the Multiplicand which you would have ſecured, ſet the Units Place of the Multiplier, and write the reſt in an inverſe Or-

 der;

der; then let each Figure in the Multiplier begin to multiply that of the Multiplicand under which it ſtands, (but ſo as to have due regard to the Increaſe that would be brought thither from the foregoing Figures) and the reſt towards the left hand as in common Multiplication, ſetting the firſt Figure of every Product under the Units Place of the Multiplier, or under that of the Multiplicand, whoſe Place is to be kept in the Product, and the Sum of theſe Products gives that ſought.

Examples.

Let ,248264 be multiplied by ,725234, and ſix Figures, or five Figures, or four Figures, of the Product only required.

For ſix Fig.	*For five Fig.*	*For four Fig.*
,248264	,248264	,248264
432527,0	432527,0	43527,0
173785	17378	1738
: : 4965	: 497	49
: : 1241	: 124	12
: : : : 50	: 5	1
: : : : : 7	1	,1800
: : : : : : 1	,18005	
,180049		

But if the above Factors were multiplied at large, the Product would be as in the Margin.

```
,248264
    0,725234
: 993056
: 744792:
: 496528::
: 1241320::
: 496528:::
1737848:::::
,180049493776
```

The

The Reaſon of the preceeding Rule may be ſhewn after this manner; let a, b, c, d, *&c.* repreſent the *Primes*, *Seconds*, *Thirds*, *Fourths*, *&c.* reſpectively of any Multiplicand, (as 24826, *&c.* where $a=2$, $b=4$, $c=8$, $d=2$, *&c.*) and e, f, g, h, *&c.* the *Primes*, *Seconds*, *Thirds*, *&c.* of the Multiplier (ſuppoſe ,7252, *&c.* then $e=7$, $f=2$, $g=5$, *&c.* here then the Multiplicand will be repreſented thus, $\frac{a}{10}+\frac{b}{100}+\frac{c}{1000}+\frac{d}{10000}$ *&c.* and the Multiplier thus, $\frac{e}{10}+\frac{f}{100}+\frac{g}{1000}+\frac{h}{10000}$ *&c.* or for brevity ſake put $n=10$, and the Terms when placed in order ſtand thus,

$$\frac{a}{n}+\frac{b}{n^2}+\frac{c}{n^3}+\frac{d}{n^4}\ \&c.$$

$$0+\frac{e}{n}+\frac{f}{n^2}+\frac{g}{n^3}+\frac{h}{n^4}\ \&c.$$ and

the Product is as follows;

:	:	:	$+\frac{ha}{n^5}$	$+\frac{hb}{n^6}$	$+\frac{hc}{n^7}$	$+\frac{hd}{n^8}$
:	:	$\frac{ga}{n^4}$	$+\frac{gb}{n^5}$	$+\frac{gc}{n^6}$	$+\frac{gd}{n^7}$	
:	$\frac{fa}{n^3}$	$+\frac{fb}{n^4}$	$+\frac{fc}{n^5}$	$+\frac{fd}{n^9}$		
$\frac{ea}{n^2}$	$+\frac{eb}{n^3}$	$+\frac{ec}{n^4}$	$+\frac{ed}{n^5}$			

But if the Multiplier is placed under the Multiplicand by our Rule, they will ſtand thus,

$$\frac{a}{n}+\frac{b}{n^2}+\frac{c}{n^3}+\frac{d}{n^4},\ \&c.$$

$$\&c.\ \frac{h}{n^4}+\frac{g}{n^3}+\frac{f}{n^2}+\frac{e}{n}+0$$ *Units Place.*

And from hence 'tis apparent, the Sum of the Indexes of every two Terms that ſtand under one another, is exactly equal to the Index of that Term

Term of the Multiplicand under which is (o) the Place of Units, *viz.* n^5 the Place in this Example to be kept; and therefore in multiplying the Factors thus placed, you need only begin with that Term of the Multiplicand which is above the multiplying Term of the Multiplier; for thereby it is evident you'll have every Term of the Product to the n^5*th* Place, in the same manner as if they are multiplied in their natural order; which will appear more evident by comparing the above Example (at large) with the following one (contracted.)

$$\frac{a}{n}+\frac{b}{n^2}+\frac{c}{n^3}+\frac{d}{n^4},\ \&c.$$

$$\&c.\ \frac{h}{n^4}+\frac{g}{n^3}+\frac{f}{n^2}+\frac{e}{n}+0\ \textit{Units Place.}$$

$$\frac{ea}{n^2}+\frac{eb}{n^3}+\frac{ec}{n^4}+\frac{ed}{n^5}$$

$$\frac{fa}{n^3}+\frac{fb}{n^4}+\frac{fc}{n^5}$$

$$\frac{ga}{n^4}+\frac{gb}{n^5}$$

$$+\frac{ha}{n^5}$$

Division of Decimals.

PROP. V.

Two Decimal Fractions being given, or two Quantities composed of Integers and Decimals, to find the Quotient arising from dividing the one by the other.

If there are more Figures in the Dividend than in the Divisor, divide them after the same manner as you do in whole Numbers; if not, supply the Defect by annexing Cyphers to the former; after which, prosecute the Division, and cut off from the Quotient as many Figures for Decimals, as there are Decimals in the Dividend more than

the Divisor; but when this cannot be done, you must prefix a sufficient Number of Cyphers to the Quotient to make good the Defect, and place the decimal Point on the left hand, so shall you have the Answer, if there is no Remainder; but when that happens, you may continue the Quotient at pleasure, by constantly annexing a Cypher to each Remainder.

Example I. Let ,13975 be divided by 43, the Work is as follows;

```
43) ,13975 (325
      129
     -----
      107
       86
     -----
      215
      215
     -----
        0
```

Now because there are five Decimals in the Dividend, and none in the Divisor, there must be five Decimals in the Quotient, which is therefore ,00325.

Examp. II. Let 5,29125 be divided by 4,25 then we have the following Process.

```
4,25) 5,29125 (1,245
       425
      -----
      1041
       850
      -----
      1912
      1700
      -----
       2125
       2125
      -----
          0
```

The

The Quotient being 1245, and because there are five Decimals in the Dividend, and but two in the Divisor, the Quotient by the Rule will have three Decimals; and from thence it is 1, 245.

Scholium. When the Terms in Division each consist of many places, and but a few Terms of the Quotient required, the Work may be shortned after the same manner as before in Multiplication: For all those Terms of the Multiplier, which when multiplied by the Figure last put in the Quotient, would give an Index more distant from Unity, than the Index of the lowest Place required in the Quotient, I say all such may be omitted by only considering the Increase to be brought from them to the first of the retain'd Figure: This will be plain by an *Example.* Let it be required to divide ,743268 by ,54213 and only to have six Figures in the Quotient; the Operation as under.

```
,5 | 4 | 2 | 1 | 3) ,743268 (1,37102
                     54213
                    ------
                    201138
                    162639
                    ------
                     38499
                     37949
                    ------
                       550
                       542
                       ---
                         8
```

Having now given the fundamental Rules of Decimal Arithmetick, it remains only that we add to what has been said, *the Method of reducing any Decimal to the known Parts of Measure, Weight or Coin, and that is performed by the following Rule.*

Multiply

Multiply the Decimal given by a Number which is equal to the multitude of Parts in the Denomination required, that makes an Integer of the Decimal given; then cutting off from the Product as many Figures from the right hand as is equal in Number to thoſe of the Decimal given, ſo ſhall thoſe on the left hand of the ſeparating Point be the Anſwer, and the Remainder (if any) a Decimal in that Denomination.

What are the Values of ,84375 of a *Pound Sterling*, of ,75 of a *Yard*, of ,754 of a *Year?*

```
 ,84375
      20 Shillings in a Pound.
 --------
16,87500
      12  Pence in a Shilling.
 --------
   1750
   875
 --------
10,500
     4  Farthings in a Penny.
 --------
 2,000
```

```
,75
  3 Feet in a Yard.
-----
2,25
,12 Inches in a Foot.
-----
3,00
```

```
 754
  13 Months in a Year.
-----
9,802
   4 Weeks in a Month.
-----
3,208
   7 Days in Week.
-----
1,456
```

So that the firſt is 16 *s.* 10½ *d.* the ſecond 2 *Feet* 3 *Inch.* the third 9 *Months* 3 *Weeks* $1,\frac{45}{100}$ *Days.* *But to find the Value of the Decimal of a Pound Sterling, the following Rule is ſufficient*, and very ready in Practice.

If

If the Figure in the Place of Seconds be above 4, add 1 to twice the Figure in the Prime's Place, ſo you have the Shillings. *Secondly*, The Exceſs of the Figure in the Seconds Place above 5, or the Figure in the Seconds Place if under 5, multiplied by 10, and added to the Thirds, is Farthings, which are eaſily reduced to Pence ; but if the Farthings exceed 12, ſubſtract one from them, and 2 when they exceed 37, ſo have you the neareſt Value (in *Engliſh* Money) of the Fraction given.

Mr. *Malcolm* and *Hill* direct to ſubſtract 1, when the Farthings exceed 23, and the firſt bids ſubſtract 2 when they exceed 40, both which are deficient.

The Invention of this Rule is as follows. Put p, s, t, for the Figures ſtanding in the firſt, ſecond, and third Places of the Decimal reſpectively, that is, p is the Primes, s the Seconds, and t the Thirds; alſo put S the Shillings, and F the Farthings ſought.

Then, $\frac{p}{10}+\frac{s}{100}+\frac{t}{1000}$ is the Fraction given, but that by Suppoſition is equal to S Shillings plus F Farthings; but S Shillings is equal to $\frac{S}{20}$*ths* of a Pound, and F Farthings is equal to $\frac{F}{960}$*ths* of a Pound ; hence have we this Equation,

$\frac{P}{10}+\frac{S}{100}+\frac{t}{1000}=\frac{S}{20}+\frac{F}{960}$, which multipled by 20, gives $S+\frac{F}{48}=2p+\frac{S}{5}+\frac{t}{50}$: Now s divided by 5, muſt either give in the Quotient 1 or 0, (for it can't exceed 9) therefore put q for 1 or 0, (the Quotient) and r the Remainder, that is, let

$\frac{s}{5}=$

$\frac{s}{5}=q+\frac{r}{5}$, then $S+\frac{F}{48}=2p+q+\frac{r}{5}+\frac{t}{50}$, thence $S=2p+q$ the Shillings, consequently $\frac{F}{48}=\frac{r}{5}+\frac{t}{50}$, or $F=\frac{480r+48t}{50}=\frac{240r+24t}{25}$, and this may be reduced to $F=10r+c-\frac{10r+c}{25}$: Now if $10r+c$ exceed 12, then $\frac{10r+c}{25}$ is greater than $\frac{1}{2}$, and so $10r+c-\frac{10r+c}{25}$ is nearest equal to $10r+c-1$; but if $10r+c$ exceed 37, then $\frac{10r+c}{25}$ is greater then $1\frac{1}{2}$, and consequently $10r+c-\frac{10r+c}{25}$ is nearest to $10r+c-2$; from whence the Rule and Correction are evident. Q. O. E.

Rules similar to this might be invented from these Principles, to value the Decimal of any other Integer, but I cannot say whether or no with equal Success in Simplicity of Expression. Seeing that the Multiplication is often performed with greater ease than Division, especially when the Divisor consists of many Figures, therefore before we finish this Chapter of Decimals, it will not be amiss to shew how such Divisors may be converted into equivalent Multipliers, which is done by this Rule.

Let Unity with a competent Number of Cyphers (as Decimals) annext, be divided by the Divisor given, and the Quotient will be the Multiplier sought: for let Q be the Quantity divided by d, and m such a Quantity that multiplying Q this Product and the former Quotient, may be equal; hence we have this Equation, $\frac{Q}{d}=m\times Q$,

or

or $m\frac{1}{d}$, or thus, $\frac{Q}{d}=Q\times\frac{1}{d}$, hence $\frac{1}{d}$ turned into a Decimal, is the Multiplier sought.

We shall illustrate this from the Divisors for reducing the Content of any Body in cubick Inches to Gallons.

$$\text{Then}\left\{\begin{array}{l}\frac{Q}{282}=\textit{Ale Gall.}\\ \frac{Q}{231}=\textit{Wine Gall.}\\ \frac{Q}{2150{,}42}\ \textit{Malt Bush.}\end{array}\right\}\text{in } Q.$$

$$\text{Whence}\left\{\begin{array}{l}m=\frac{1}{282}=,003546 \text{ the Multiplier for Ale.}\\ m=\frac{1}{231}=0043289 \text{ the Multiplier for Wine.}\\ m=\frac{1}{2150{,}42}=0004651 \text{ the Multiplier for Malt.}\end{array}\right.$$

CHAP. II.

Of the Square Root.

Defin. I. THE square Root of any Number or Quantity is that which multiplied by it self, shall produce the said Number or Quantity. Thus, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, *&c.* are respectively the square Root of 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, *&c.* because 1x1, 2x2, 3x3, 4x4, 6x6, *&c.* produce those Numbers; hence this Table of simple ple Roots and Squares:

Roots.	1	2	3	4	5	6	7	8	9	*&c.*
Square.	1	4	9	16	25	36	49	64	81	*&c.*

Defin. II. The square Roots of Numbers are of two Sorts, *simple*, when they consist of one Figure only;

only; and *compound*, when of more than one; therefore the Square Root of every Square Number of less than three Figures, is Simple, and may be had from the above Table; but to obtain those which are compounded, we must premise the following *Lemmas*.

Lemma.

If n *be the Number of Integral Figures in the Square Root of any Number, then* $n+1$ *shall be the Number of Figures in the Square Root of the same Number when two Figures are annext to it.*

For let Q be the Number whose Square Root q consists of n Figures, this when two Figures are annext will be 100Q, but since $q=\sqrt{Q}$ by Supposition, therefore $100q2=100Q$, and therefore $10q$ is the Square Root of 100Q, but there is n Figures in q, and therefore $n+1$ Figures in $10q$, or in the Square Root of Q, when two Figures are annext. *Q.E.D.*

Corol. 1. Hence if any Number be divided into Parts of two Figures, each by beginning at the Right-hand, and so on to the Left, the Square Root will consist of as many Integral Figures as are in the Number of Points more by one; that is, if P be the Number of Points, then $P+1$ is the Number of Integral Figures in the Root: Also the Square Root of the first Division to the Left-hand, if a Square Number, or the Root of the next less, if otherwise, will be the first Figure of the Root.

Thus let the Number be 1743268 this divided as above ſtands thus, 1.74.32.68; from whence it appears there will be four Integral Figures in the Root, and the firſt Figure will be one.

It is common to point the above Number thus, 1743268, and then the Points ſtand under the laſt Figure that is brought down at each Operation, and we have followed this method to prevent Confuſion to ſuch as have been uſed to the ſame, tho' the firſt ſeems moſt natural; but 'tis probable the laſt was choſe to avoid Miſtakes when the Root of a mixt Number was ſought, that conſiſted of Integers and Decimals, becauſe of the Reſemblance of the Point (.) and the Decimal Mark (,)

Corol. 2. By the ſame way of Reaſoning it may be ſhewn, that if q the r Root of any Number Q conſiſt of n Integral Figures, then the Root of the ſame when r Figures are annext to it, ſhall conſiſt of $n+1$ Figures. And therefore univerſally if any Number whoſe r Root is ſought, be divided from the Right-hand into Parcels, each of which has r Number of Figures, the Root ſhall conſiſt of as many Figures *plus* one, as are the Number of ſeparating Points. Thus the by-quadrate Root of 1726854396, ſhall have two Figures in it, becauſe when divided, it ſtands thus 17-2685-4396, and 2 is the firſt Figure of the Root.

Corol. 3. Therefore if any Square Number (whoſe Root is required) be pointed, by placing one Point over the Units, another over the Hundreds, and ſo on for every other Place to the left

left hand, then from the above Lemma, the Root will consist of as many Figures as in the Number of Points placed above the same.

LEMMA II.

If a^2 denote any square Number wanting the two last Figures, and b be put for them, then shall (e) the remaining Figure of the Root be $\frac{b}{20a+e}$, or $e = 2a$) $\frac{b}{10}$ (nearly: for $100a^2+b$ is the given square Number, whose Root is supposed $10a+e$, therefore $\overline{10a+e}|^2 = 100a^2+20ae+ee = 100a^2+b$, whence $e = \frac{b}{20a+e}$, but $10a$ is much greater than e, whence $\frac{b}{20a}$, or $\frac{\frac{b}{10}}{2a}$ is nearly equal to e: Consequently if $\frac{b}{10}$ be divided by $2a$, so that e the Quotient being annexed on the right hand $2a$, and that multiplied by e, the Product may be equal to b, then is $10a+e$ the Root sought; but if no such Number (e) can be found, take the next less, and place it on the right hand $2a$ as before, the Product of which, and the Divisor so found, must be taken from b, and the Remainder shews how much the said Number exceeds an exact Square Number. From hence, and the first Lemma, we deduce this Rule.

To find the square Root of any given Number.

Let the Number whose Root is sought be pointed, beginning at the Units place, and so on towards the right hand, placing a point over every other Figure: Then seek among the simple Roots, that, whose Square is equal, or the next less, to the first Period, this is the first Figure of the Root;

let it be squared and taken from the first Period, and the next Period annexed to the Remainder, so have you what is called a *Resolvend*; let this wanting the Units place (which is equivalent to $\frac{b}{10}$) be considered as a Dividend, and twice the first Figure of the Root the Divisor, the Quotient of this Division is the second Figure of the Root, which being annext to the said Divisor, the Product thereof by the said second Figure, must be taken from the Resolvend, and another Period annexed to the Remainder, will form a new *Resolvend*; this wanting the Units Place, is again to be considered as a Dividend, and twice the Figures already found of the Root the Divisor, the Quotient of which is the third Figure of the Root, which being (as before) annext to the last Divisor, and the whole multiplied thereby, the Product taken from the last Resolvend gives a Remainder, to which another Period must be annext for a new Resolvend; and with this, and the Figures already found of the Root, you must proceed as before, to get the fourth Figure, and so on till all the Periods are brought down: But if there should be still a Remainder, then the given Number has no exact Root, but you may approach it at pleasure, by annexing two Cyphers to every such Remainder, for a new Resolvend; this will be made plain by the following Example.

Example I. Let it be required to find the square Root of 273529, this pointed, stands thus $2\dot{7}3\dot{5}2\dot{9}$, from whence it appears (from Lemma I.) there is three Figures in the Root (if a square Number): The first Period is 27, which is not an exact square Number, but the next less is 25, whose Root 5 is the first Figure of the Root sought; then 25 taken from 27 leaves 2, to which I annex

35

35 (the ſecond Period) ſo we have 235 for the firſt Reſolvend, of which 23 is the Dividend, and 10 (the double of 5) the Diviſor; but 23 divided by 10, gives 2 in the Quotient, which is the ſecond Figure of the Root, and this annext to the Diviſor 10 gives 102, and 102 multiplied by 2 gives 204, which taken from the Reſolvend 235 leave 31, to which I annex 29 (the next Period) ſo we have 3129 for a new Reſolvend, of which 312 is the Dividend, and 104 (the double of 52) the Diviſor. Then 312 divided by 104 gives 3 in the Quotient, which annex'd to 104 makes 1043, this multiplied by 3 gives 3129, and this taken from 3129, the laſt Reſolvend, there remains (0): And becauſe all the Periods are brought down and no Remainder, we may conclude 523 is the Root ſought. See the following Work, where obſerve a denotes the firſt Figure of the Root, a' the two firſt, a'' the three firſt, &c. alſo e denotes the ſecond Figure of the Root, e' the third Figure, e'' the fourth Figures, &c. R is the firſt Reſolvend, R$'$ the ſecond, R$''$ the third, &c. ſo that $a'=10a+e$ $a''=10a'+e'$, $a'''=10''+e''$, &c. But when any of the e's is a Decimal, then the next a is only equal to the Sum of this e, and the laſt a. For in that Caſe $\overline{a+e}|^2=a^2+b=a^2+2ae+ee$, or $e=\sqrt{\dfrac{b}{2a+e}}$ and $a'=a+e$, which will be manifeſt from the following Example.

$$\dot{2}7\dot{3}5\dot{2}9\ (5=a$$

$$25$$

$$\left.\begin{array}{r} 2a=10 \\ 20a+e=102 \end{array}\right\} \begin{array}{l} 235=R\ (2=e\ \therefore\ a'=52 \\ 204=e\times\overline{20a+e} \end{array}$$

$$\left.\begin{array}{r} 2a'=104 \\ 20a'+e'=1043 \end{array}\right\} \begin{array}{l} 3129=R'\ (3=e'\ \therefore\ a''=523 \\ 3129=e'\times\overline{20a'+e'} \end{array}$$

$$0000$$

Examp. II. What is the ſquare Root of 4786954; this being pointed ſtand thus $\dot{4}78\dot{6}9\dot{5}4$, ſo that by Lemma Iſt, there will be four Figures in the integral Part of the Root, the Operation as follows.

$$\dot{4}7\dot{8}6\dot{9}5\dot{4}$$

$$
\begin{array}{r|l}
 & \dot{4}7\dot{8}6\dot{9}5\dot{4}\ (2=a \\
 & \underline{4=a^2} \\
\left.\begin{array}{r} 2a=4 \\ 2a+e=41 \end{array}\right\} & \begin{array}{l} 78=R\ (1=e\ \therefore\ 10a+e=21=a' \\ \underline{41=\overline{2a+e}\times e} \end{array} \\
\left.\begin{array}{r} 2a'=42 \\ 2a'+e'=428 \end{array}\right\} & \begin{array}{l} 3769=R'\ (8=e'\ \therefore\ 10a'+e'=218=a'' \\ \underline{3424=\overline{2a'+e'}\times e'} \end{array} \\
\left.\begin{array}{r} 2a''=436 \\ 2a''+e''=4367 \end{array}\right\} & \begin{array}{l} 34554=R''\ (7=e''\ \therefore\ 10a''+e''=2187=a''' \\ 30569=\overline{2a''+e''}\times e'' \end{array} \\
\left.\begin{array}{r} 2a'''=4374 \\ 2a'''+e'''=4374{,}9 \end{array}\right\} & \begin{array}{l} 3985{,}00=R'''\ ({,}9=e'''\ \therefore\ a'''+e'''=2187{,}9=a'''' \\ 393741=\overline{2a'''+e'''}\times e''' \end{array} \\
\left.\begin{array}{r} 2a''''=4375{,}8 \\ 2a''''+e''''=4375{,}81 \end{array}\right\} & \begin{array}{l} 475900=R''''\ ({,}01=e''''\ \therefore\ a''''+e''''=2187{,}91=a \\ \underline{437581=\overline{2a''''+e''''}\times e''''} \end{array} \\
 & 3{,}8319 \text{ the Remainder.}
\end{array}
$$

And if to this Remainder two Cyphers were added, and a''' doubled as before, we might proceed to find another Figure, and so another, till as many Figures were had as would be sufficient; but after the Root is found pretty near, a good many of the remaining Figures will be had by dividing by the last Divisor; thus in the present

 Case,

if 3,8319 be divided by 4375,8 the Quotient, 0,00876 are the Figures to be annext to 2187,91; so that the square Root of 4786954 is 2187,910876 nearly, which was required: Or having obtained the Root pretty near, you may treble the Figures of the Root already found by this Rule; let Q be the Number whose Root is sought, and r denote any of the above Remainders which we expressed by R, R′, R″, R‴, *&c.* with remaining Figures annext; also put q for the Figures already found of the Root, when the Remainder r was got, then the Figures still wanting of the Root will be express'd by $\frac{2rq}{4q^2+r}$, and in this if r be so small in respect of $4q^2$ that it may be neglected, then this Expression becomes $\frac{r}{2q}$; the same with what we have above in Words. But to save the trouble of squaring $2q$ in order to get $4q^2$, you need only substract r from Q, and the Remainder is q^2. The Reason of this is omitted, as being easily deduced from Dr. *Hally's Rational Theorem for Pure Powers*, wherein he shews if b denotes how much any Number exceeds the mth Power of a, that the mth Root thereof shall exceed a by $\frac{ab}{ma^m+\frac{m-1}{2}b}$, or that $\sqrt[m]{a^m+b}=a+\frac{ab}{ma^m+\frac{m-1}{2}b}$ *proximè*.

If the Number given was a decimal Fraction, or composed of Integers and Decimals, then the Units place being found, let the Integers (if any) be pointed as before, and for the Decimals a point must be put over every other Figure towards the right hand, beginning at the place of hundreds, or seconds; after which the Method of Operation is the

the same as if the whole were Integers, only remember to separate as many Figures for Decimals in the Root, as are the Points over the Decimal Parts of the Number it self. This will be plain from an Example. Let it be required to find the Square-Root of 4226,754695, this pointed stand thus $4\dot{2}2\dot{6},7\dot{5}4\dot{6}9\dot{5}$, from whence it appears there will be two integral, and three Decimal Figures in the Root. The Operation follows.

$$
\begin{array}{rl}
 & 4226{,}754695\ (6=a \\
 & 36=a^2 \\
 & \overline{} \\
2a=12 & \\
2a+e=125 & \Big\}\ 626=R'\ (5=e\ \therefore\ 10a+e=65=a' \\
 & 625=\overline{2a+e}\times e \\
2a'=130, & \\
2a'+e'=130{,}0 & \Big\}\ 175=R'\ ({,}0=e'\ \therefore\ a'+e'=65{,}0=a'' \\
 & 000=2a'+e'\times e' \\
 & \overline{} \\
2a''=130{,}0 & \\
2a''+e''=130{,}01 & \Big\}\ 17546=R''\ ({,}01=e''\ \therefore\ 00a''+e''=65{,}01=a''' \\
 & 13001=\overline{2a''+e''}\times e'' \\
 & \overline{} \\
2a'''=130{,}02 & \\
2a'''+e'''=130{,}023 & \Big\}\ 454595=R'''\ ({,}003=e'''\ \therefore\ a'''+e'''=65{,}013=a'''' \\
 & 390069=2a'''+e'''\times e''' \\
 & \overline{} \\
 & {,}064526=\text{Remainder.}
\end{array}
$$

And

And here if the Remainder ,064426 be divided by 130,02, the last Divisor, the Quotient ,0004955 is nearly the remaining Part of the Root, which therefore is 65,0134955 : but we shall take this Opportunity to illustrate what was said of tripling the Figures already found of the Root; in this Example $Q=4226{,}754695$, and having got two Figures of the Root, (*viz.* 65) we find $R'=,175$, to which add the other Figures, and we have $r=1{,}754695$, whence $Q-r=q^2=4225$, as is evident, (for $q=65$) hence $4q^2=16900$, therefore $4q^2+r=16901{,}754695$, and $2rq=130\times1{,}754695$ $=228{,}11035$, therefore $\frac{228{,}110350}{16901.754695}$ is the remaining part of the Root nearly; but in this Division we need only six Figures of each, the Operation is is as hereunder,

```
1|6|9|0|1|8) 228,110 (,013496
             169018
             ------
              59092
              50705
             ------
               8387
               6761
             ------
               1626
               1521
             ------
                105
             ------
                102
             ------
                  3
```

CHAP.

CHAP. III.

The Construction of the Sliding-Rule, with the Use thereof, and the Reason of the Method of Operation thereon Demonstrated.

SINCE, as we observed before, the Practice of Gauging almost entirely depends on the Knowledge of the Sliding-Rule, it must be of great Importance to the Gentlemen of the Excise to be acquainted, not only with the Method of Operation thereon, but the Reason thereof: 'Tis true indeed, so many have already wrote on this Subject, that it will seem superfluous to attempt any thing of this kind at present; but besides observing that those Authors, (especially the most modern) not being sufficiently acquainted with the Theory, have chiefly confined themselves to the Practice; so what they have done even in that is so irregular, that I fear few of their Readers have a sufficient acquaintance with the Nature and Properties of this valuable Instrument. Therefore in what follows I design to trace it from its Original, and that in so regular and plain a Manner, as to render the Construction and Use thereof so easy, as to come within the Power of those but indifferently versed in Mathematical Knowledge: And in order to this, in the first place, the Nature and Properties of Logarithms must be first explained, because the principal Lines on the Sliding-Rule are logarithmical, for which purpose we shall lay down the following Doctrine.

SECT. I. *Of the Properties of Powers and Exponents.*

In Specious Computations, every one is at Liberty to chuse after what manner he will signify the

the Sum, Difference, Product or Quotient of any Quantities *a*, *b*, *c*, *d*, *&c.* for it is a matter of Indifference by what signs or Manners they are represented, so that we are constant in our Practice, and always follow the same Rule; for thereby we shall ever be able to go back from any Sum or Product to its component Factors, which is sufficient in all kinds of computation: It is common to express the Sum of the above Quantities thus, $a+b+c+d$ *&c.* and their Product thus, $a \times b \times c \times d$ *&c.* or for brevity sake, some omit the mark (x), and signify their Product thus, $abcd$, by joining the several Factors together, as in forming Words from Letters. From this last way of expressing a Product, if the several Factors were the same, (viz. a), then those Products would be thus $aaaa$, *&c.* which are called the Powers of the Quantity a; the first being a, the second aa, the third aaa; that is any Power of a, expressed after this manner, is signify'd by as many a's set together, as is the Number expressing that Power. But it is found troublesome to express large Powers of Quantities after this manner, hence some who had no occasion to consider any Powers beyond the 5th or sursolid, chose to denote the several inferiour Powers by a particular Letter or Letters adjoined to each, as the Square of A, was denoted by A*q*; the Cube of A, by A.*c*; the fourth Power of A, by A.*qq*; the fifth Power of A, by A.*qc*, and so on: but this, as observed already, is only fitted for small Powers, so that the first Method was preferable. But the most illustrious *Newton* has introduced a new Method for this purpose, both more easy and more general than the others, extending to Roots as well as Powers, and that how high soever; the Properties of which we shall deliver in the following Propositions, after which the

the Reason of the Method of Operation by Logarithms will be easy.

Definition. Any Power of a Quantity a, is expressed by the said Quantity a, with a Number called *the Index*) placed above towards the right hand, expressing the Place it has from Unity in a Geometrick Series, whose first Term is 1, and Ratio the Root it self a, as 1, a, aa, aaa, $aaaa$, $aaaaa$, *&c.* thus for Example, $aaaa$, being the 4th Term from 1, is expressed by a^4; also $aaaaa$, being the 5th Term from 1, is expressed by a^5, and so of others.

PROP. I.

The Index of the Product of any 2 Powers of the same Quantity, is equal to the Sum of the two Indexes of the Factors, viz. $a^m \times a^n = a^{m+n}$.

The Indexes both here and in what follows, are supposed Integers.

For by what was said above $a^m = aaaaa$, *&c.* till the Number of a's is m, and $a^n = aaaaa$, *&c.* till the Number of a's is n; whence $a^m \times a^n = aaaaaaaaaa$, *&c.* till the Number of a's is $m+n$, which Power of a (by the Definition) is expressed a^{m+n}. *Q. E. O.*

PROP. II.

The Index of the mth *Power of* a, *viz.* a^m, *raised to the* nth *Power, viz.* $\overline{a^m}|^n$, *is equal to* $m \times n$.

For, by the Definition, $\overline{a^m}|^n = a^m \times a^m \times a^m \times a^m$ *&c.* till the Number of Factors is n, but $a^m \times a^m \times a^m \times a^m$ *&c.* is, by the last, equal to $a^{m+m+m+m}$ *&c.* and the Number of m's being n, 'tis evident $m+m+m+m$ *&c.* $= m \times n$, whence $\overline{a^m}|^n = a^{m \times n}$. *Q. E. O.*

PROP. III.

The Index of the Quotient of one Power of any Quantity, viz. a^m, *divided by another Power of the same Quantity, viz.* a^n, *is equal to the Index of the Divisor* n, *taken from that of the Dividend* m; *that is* $\frac{a^m}{a^n} = a^{m-n}$.

For put $\frac{a^m}{a^n} = a^x$, and let the Equation be multiplied by a^n, then, by *Proposit.* I. we have $a^m = a^{m+x}$; therefore, since the Roots are the same and the Powers equal the Indexes must be so too, whence $m = n + x$, or $x = m - n$. Q. E. O.

Cor. I. From this Proposition it is plain $a^o = 1$ for if $m = n$, then $x = o$, but $\frac{a^m}{a^m} = 1$, therefore $a^x = a^o = 1$.

Cor. II. Also from hence it follows $\frac{1}{a^n} = a^{-n}$; for $\frac{a^m}{a^n} = a^{m-n}$, hence if $m = o$ then $\frac{a^o}{a^n} = a^{-n}$ but $a^o = 1$, thence $\frac{1}{a^n} = a^{-n}$.

SECT. II. *Transition to Logarithms.*

From these Propositions we are enabled to demonstrate the Method of Operation by Logarithms: For 'tis most evident, if between 1 and any Number (suppose 10,) an infinite, or only a very great Number of Proportionals 1000000 be taken, thereby forming a Geometrical Series, some of them

them must approach very near to the natural Numbers, 2, 3, 4, 5, 6, 7, 8, 9; and by continuing this Series, the subsequent Terms will also approach very near the following Numbers 11, 12, 13, 14, &c. till the whole Scale of Numbers to any desired one, (as 100000 the Extent of the largest Table of Logarithms) is compleat: Hence, thro' the natural Numbers themselves do not form a geometrick Series, yet on the above Supposition, they will obtain Places, very near, in such a Series; and then,

Definition. *The Place from 1, which any of them has in that Series, which is ever the Index of the second Term, is called it Logarithms.*

Thence it follows that Logarithms may be said to be the *Exponents* of Ratio's.

Having given this Idea of the Nature of Logarithms, we should proceed to shew how they might be computed: But since every body at present may be supposed furnish'd with Tables, we shall from thence take such as will enable us to find the Divisions on the Sliding-Rule: But that nothing might be omitted to render this Treatise compleat, we shall add, at the End, Rules for computing them, as delivered by the incomparable Dr. *Halley*, with some others, ready in Practice.

Now let a denote the first Term of the infinite Scale of Proportionals, which is taken betwixt 1 and 10; and let us denote the Logarithm of any Number by the Number it self with L: prefixt: Thus the Logarithm of 2 we express by L: 2; the Logarithm of 3 by L: 3; the Logarithm of 4 by L: 4, &c. then 'tis manifest this infinite Scale of Proportionals will stand thus, $1, a, a^2, a^3, a^4,$ $\ldots\ldots 2=a^{L:2} \ldots\ldots 3=a^{L:3} \ldots\ldots 4=a^{L:4}$ $\ldots\ldots$ &c. $10=a^{10000}$, &c.

PROP. I.

The Sum of the Logarithms of any Numbers, is equal to the Logarithms of the Product of those Numbers.

For let the two Numbers be denoted by A and B, then I say, $L:A+L:B=L:A\times B$.

By what was just observed it is plain, $A=a^{L:A}$, and $B=a^{L:B}$, therefore $A\times B=a^{L:A+L:B}$ (*per* Prop. I. *of the last Section*,) but (*per Definition*) the place which any Number obtains in the Series is its Logarithm, and that is ever the Index of *a* the second Term; whence $L:A+L:B=L:A\times B$. *Q. E. O.*

PROP. II.

The Logarithm of the Quotient of any two Numbers, is equal to the Logarithm of the Divisor, taken from that of the Dividend,

that is, $L:\frac{A}{B}=L:A-L:B$.

For if the above Value of A be divided by that of B, we have $\frac{A}{B}=a^{L:A-L:B}$, by *Proposit.* III. foregoing. Whence by what was already observed, $L:\frac{A}{B}=L:A-L:B$. *Q. E. O.*

PROP. III.

The Logarithm of any Power of a Quantity is equal to the Logarithm of the Quantity it self multiplied by its Index, that is $L:A^m=m\times L:A$.

For since $A=a^{L:A}$, 'tis manifest, (*by* Prop. II. *of the last Section*) $A^m=a^{m\times L:A}$, whence 'tis evident $m\times L:A=L:A^m$. *Q. E. O.*

Cor.

Cor. From this Propoſition it appears, if there are two Scales of Logarithms, in which the Logarithm of any Number in the one is double of the Log. of the ſame Number in the other, then the Logarithms of all Numbers taken from the former, ſhall be the Logarithms of the Square of the ſame Numbers taken from the latter: For we have ſhewn that $L : A^m = m \times L : A$, therefore if $m=2$ then $L : A^2 = 2 \times L : A$. Again, if there are two Scales, or ſorts of Logarithms, where the Logarithm of every Number in the one is triple the Logarithm of the ſame Number in the other, then the Logarithms of all Numbers in the former ſhall be the Logarithms of the Cubes of the ſame Numbers taken from the latter: For in the laſt by making $m=3$ we have $L : A^3 = 3 \times L : A$, and A is put indefinitely for any Number. Theſe Corollaries muſt be well conſidered, becauſe on them depends the Reaſon of the Operation of a ſingle Radius (of the Rule) ſliding againſt a double or triple one, by which we find a fourth Proportional to theſe three Quantities, $Q^2 : R^2 : S$: or to theſe three, $Q^3 : R^3 :: S$: when the Value of Q, R and S are only given. Having now demonſtrated the method of the Operation by Logarithms, which was never generally done before, we ſhall proceed to the Conſtruction of the Sliding-Rule.

Sect. III. *The Conſtruction of the Sliding-Rule.*

The Rule we are to conſtruct, is in the Form of a Parallelopipedon, on three Faces of which are three graduated Sliders, ſliding by ſix graduated Lines on the Rule; the firſt Face is repreſented by Fig. 1. (*Plate* 1.) the ſecond is repreſented by

Fig. 2. the third by Fig. 3. and the fourth by Fig, 4. of each of theſe in their order as they are marked A, B, C, D, E, MD, SL, SS, and N. The Lines graduated on the firſt Side, are four Lines of Numbers, or Logarithms, or rather three, becauſe the Sliding-piece marked B has two Lines, each divided in the ſame manner: A is the firſt of theſe Lines; to delineate which, let the Sector be opened till 10, 10, on the Line of Lines be equal to the Diſtance betwixt one and ten on A; then ſince the Logarithms of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, are 0, 301, 477, 602, 699, 778, 845, 703, 954, 1,000 reſpectively from the Line of Lines on the Sector thus opened, take theſe Numbers, and lay them reſpectively from 1 along the ſame Line A, ſo ſhall you have the Points anſwering the Numbers, whereof the ſaid Diſtances are Logarithms; the ſame Diſtances laid from 10, (which on the Rule is only marked 1) will give the other Diviſions on this Line, it containing its Radius twice; then ſince the firſt of theſe Diſtances, (*viz.* betwixt 1 and 2) is divided into 50 Parts, let the Logarithms of $1\frac{1}{50}$, $1.\frac{2}{50}$, $1.\frac{3}{50}$, $1.\frac{4}{50}$, &c. be taken from the Table of Logarithms, and the diſtance anſwering thoſe Log. taken from the Sector opened as before, and laid from 1 along A; ſo ſhall you have all the Diviſions of the Space 1, 2. Again the Space 2,3; 3,4; 4,5 are each divided into 20 Parts; wherefore, let the Logarithms of $2.\frac{1}{20}$. $2.\frac{2}{20}$, $2.\frac{3}{40}$, $2.\frac{4}{40}$, &c. be alſo taken from the Sector open'd as before, and laid from 1 along A, ſo will you have all the Diviſions to 3, then the Logarithms of 3, $3\frac{1}{20}$, $3\frac{2}{30}$, $3\frac{3}{20}$, &c. to 4; then the Log. of 4; $4.\frac{1}{20}$, $4.\frac{2}{20}$, $4.\frac{3}{20}$, $4,\frac{4}{20}$, &c. to 5, and the diſtances correſponding to theſe ſeveral Logarithms, being ſucceſſively taken from the Sector opened as before, and laid from 1 towards A, will give you all the Diviſions betwixt 2 and 5 in both

Ra-

Radius's. Then since the remaining Spaces, *viz.* 5,6, and 6,7, and 7,8, and 8,9, and 9,10, are each only divided into 10 Parts; let the Logarithms of $5\frac{1}{10}$, $5\frac{2}{10}$, $5\frac{3}{10}$, *&c.* to 6 be found; also those of $6\frac{6}{10}$, $6\frac{2}{10}$, $6\frac{3}{10}$, *&c.* to 7, and so on till you have found all the Logarithms of the Series, $5\frac{1}{10}$, $5\frac{2}{10}$, $5\frac{3}{10}$ . . . . 6, $6\frac{1}{10}$, $6\frac{2}{10}$, $6\frac{3}{10}$ . . . $7\frac{1}{10}$, $7\frac{2}{10}$, $7\frac{3}{10}$, . . . $8\frac{1}{10}$, $8\frac{2}{10}$, $8\frac{3}{10}$, . . . $9\frac{1}{10}$, $9\frac{2}{10}$, $9\frac{3}{10}$, . . . 10; then let the Distance corresponding to each of these Logarithms be taken from the Sector (still open'd as before,) and laid from 1 along A, so will you have all the Divisions on this Line.

The Lines B, C, are exactly the same with A, so nothing further is necessary to be said on them, but only instead of A in the above Construction, to read B or C; the Line D has a single Radius exactly double to A, B, *&c.* so that the Distance from 1 corresponding to every Number on this, is just double of the Distance from 1 corresponding to the same Number on any of them.

The Line E has a triple Radius, each of which is a third Part of the Radius D, so that the Distance from 1 corresponding to any Number on this Line is exactly $\frac{1}{3}$ of the Distance from 1 on the same Number on D, or two thirds of the Distance betwixt one and the same Number on any of the Lines A, B, and C; whence it is manifest from the Divisions on A, all those on B, C, D and E are immediately had. We are now come to the Line mark'd MD, which is also Logarithmetic, and the same with A, B, C, D, but inverted, and does not begin from 1 as the others do, but from 21.5042, and so on in an inverted Order, so that the Order of the Numbers on this Line are 2 (or 200) 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, .9, 8, 7, 6, 5, 4, 3, and the last Division is 2,15042, by

which dividing 215, you have 100 for the Quotient, which is the ſame with the laſt Diviſion on A, B, C, and D; by which means it will be found that the Point 1 will be exactly over againſt the Points on A, marked with M B, and a Braſs Pin, which correſponds with the Number 2150,42, or 215,042, or 21,5042: the firſt of theſe being the cubic Inches in a Buſhel of Malt, for if from the Logarithm of 10, or 100, the Logarithm of 2,15042, or 21,5042 be taken, the Remainder ſhews the Diſtance of *one* from the end of the Line MD, which is exactly the ſame with the Diſtance of the Points MB from the end of the Line A, that being alſo equal to the Difference betwixt the Logarithm of 10 or 100, and 2,15042 or 21,5042. The Reaſon why this Line is inverted, and its beginning and ending as above, will be ſhewn when we come to the Uſe of the Sliding-Rule.

Cor. From what has been ſaid, it is plain the Sliding-Rule is analogous to a Table of Logarithms, the Diſtances of the Diviſions on A, B, C, D and E, being as the Log. of their Ratios.

Having now ſhewn how the Lines on two oppoſite Faces of the Sliding-Rule are obtained, we come to the other two; on one of which are four Lines, *viz.* a Slider with two Lines, which may be reduced to one, they being only the ſame Diviſions, continued to its two Edges, and two others on the Rule it ſelf, to operate with the Slider; the Slider itſelf mark'd N, is exactly the ſame with the Lines A, B, C, already deſcribed, ſo that what was ſaid of them is ſufficient for this. One of the fix'd Lines on this ſide is mark'd with S: L: or *Seq. ly.* whoſe office, with the aſſiſtance of the Slider, is to find the Quantity of Liquor drawn, or remaining in a Caſk not full, and

whoſe

whose Axis is parallel to the Horizon: The other fixed Line is mark'd SS, or *Seq. St.* whose Use is to find the Quantities of Liquor drawn out of a Cask when its Axis is perpendicular to the Horizon: But the Constructions of these Lines depending on Principles not yet laid down, we shall *postpone* them till we come to *Ullaging*, which is handled in the last Part.

On the other Side opposite this are 3 fixed Lines, whereon is wrote *Spheroid Variety*, &c. whose Uses are to find what is called *the mean Diameter*, or to reduce a Cask of any of the three Varieties to Cylinders; but that being of little use, and so easily done without a mean Diameter, it is not worth spending any Time on their Explication. On the Insides of the two first Sliders are placed two Lines, the the one numbred from 13 to 16, and the other from ,42 of a Gallon to 3,61 Gallons; which is only a Table to shew the Contents in Ale Gallons of all Cylinders, whose Depths are 1 Inch, and Diameter from 13 Inches to 36 inclusive, which are easily computed from the Rule following for gauging Cylinders, so that we shall not dwell any longer on them, but proceed to

Sect. IV. *The Use of the Sliding-Rule.*

And first, of Numeration, or the Method of estimating the Values of the Divisions on the Sliding-Rule, for which this is a general Rule: For the Lines A, B, C, D, MD and N; the Value of the first on the Rule may be assign'd at pleasure, either 1, 10, 100, 1000, &c. or ,1; ,01; 001, &c. and whatever that is, the first 2 is twice as much, the first 3 thrice as much, the first 4 four times as much, and so on to the second 1, which is ten times the first; the second 2, 3, 4, &c. being ten times the Value of the first 2, 3, 4, 5, 6, &c. And

where there are three 1's, or a triple Radius, the third 1, 2, 3, 4, 5, 6, *&c.* are 100 times the Value of the first, or ten times those of the second; and this may serve for the integral Divisions; but those being known, the intermediate ones are easy; for 'tis but considering into how many Parts the Space betwixt any two of the integral ones is divided, and the difference of their Values, for then the first of them is estimated by a Fraction, whose Numerator is the Difference of the Values of the adjacent Figures on each side, and its Denominator is the Number of Parts into which the Space betwixt the Integers is divided; the second Division is twice the Value of the first, the third thrice its Value, *&c.* and if to any of these be added the Value of the preceding Figure, you will have the Number to which that Division on the Rule corresponds. We will illustrate this by an Example or two on the Lines A, E. Let the first 1 at the end of the Rule on A stand for 1, then the second 2 stands for 20; now since the Space betwixt the second 2 and 3 is divided into 20 Parts, this is the Denominator of all the Divisions betwixt the second 2 and 3, and the Difference betwixt the second 2 and 3, *viz.* 30—20 is 10; so from the Rule given above, the first Division after the second 2, is $20\frac{10}{20}=20\frac{1}{2}$; the second Division $20+\frac{2\times10}{20}=21$; the third $20+\frac{3\times10}{20}=21\frac{1}{2}$, *&c.*

Let us take a second Example on the Line E, which has a triple Radius, and let the first 1 stand for 10, then the third 1, (which often is marked 10) signifies 1000; and the third 5 stands for 5000; then because the Space betwixt this 5 and the following 6 is divided into ten Parts, and the Difference of their Divisions is 1000; the first Division after the third 5, signifies $5000+\frac{1000}{10}=5000+100=5100$, the second is $5000+2\times\frac{1000}{10}=5000+200=5200$, and so of the rest.

Multi-

Multiplication by the Sliding-Rule.

Prop. I.

Two Numbers being given, it's required to find their Product by help of the Sliding-Rule.

Rule. Set 1 on B to either of the Factors on A, then against the other Factor on B is the Product on A.

For the two Numbers being represented a, b, and their Product by p, we have $a \times b = p$, or (*by Prop.* I. *of the 2d Sect.*) $L:p = L:a + L:b$; now 'tis plain, when 1 on B is removed to correspond with a on A, then the Division on A, which is opposite to b on B, shall be distant from 1 on A, by the Sum of the Distances belonging a, b; but those Distances are Logarithms of the Numbers a, b, by what hath been said above; whence the Distances of the Division on A, which is opposite b, on B shall be expressed by $L:a+L:b$; but the $L:a+L:b=L:p$ from the foregoing, whence the Number answering that Division is equal to p, the Product of a, b. Q. E. O.

Example I. Let the Product of 15 and 12 be sought. By the above Direction, set 1 on B to 12 on A, then against 15 on B, I find (by the preceeding Numeration) is the Number 1,80, the Product sought.

There may occur some Difficulties to Beginners in these Matters, for it may happen either that after 1 on B is placed to one of the Factors on A, the other Factor may not be found on B, because of the shortness of the Rule, or of its not having a sufficient Number of Radius's, or that when the second Factor is found on B, it falls beyond the Line A; in both which Cases you must divide

one or both the Terms by 10, 100, 1000, *&c.* not only till they are both to be found on their respective Lines, but also that the Division corresponding with the Number on B, may fall opposite some Division on A: and then you are to multiply the Product thence arising, by what both the Factors were divided by. The Reason of this is manifest; for the Factors being a, b, and their Product P, *viz.* $a \times b = p$, then $\frac{a \times b}{n} = \frac{p}{n}$, consequently $\frac{p}{n} \times n = p$. We will illustrate this by Examples.

Example I. Let the Product of 3 and 245 be required, 'tis plain the first 1 on B being esteemed 1, the last Division on that Line is ,100, and therefore 245 is off the Rule; but let 245 be divided by 10, and the Factor become 3 and 24,5, which by the Rule gives 73,5 for the Product; therefore because 245 was divided by 10, I multiply 73,5 by 10, and get 735 for the Product of 3 by 245.

Example II. Let the two Numbers be 400 and 350, here taking the first 1 on B for *one* 350 is off the Rule, so I divide it by 10, and have 35; but when 1 on B is set to 400 on A, (the first 1 on that Line being called 100) then 35 falls beyond the Line A, so I divide it again by 10, that is 350 is divided by 100, and against the Quotient (3,5) on B is 1400; but seeing 350 was divided by 100, I multiply 1400 thereby, and have 140000 for the Product of 350, by 400.

Division

Division by the Sliding-Rule.

PROP. II.

Two Numbers being given, 'tis required to find their Quotient by help of the Sliding-Rule.

Rule. Set 1 on A to the Divisor on B, then opposite the Dividend on B, is the Quotient on A.

The two Numbers being as above called a, b, of which b is the Dividend, and a Divisor, and the Quotient q, then $\frac{d}{a}=q$, or L : b—L : a=L : q (by *Prop.* 2. *Sect.* 2.)

But when 1 on A is set to a on B, 'tis evident the Number on A opposite b on B, shall correspond to the Excess by which the Distance from 1 to b, exceeds that from 1 to a: and these Distances being Logarithms of those Numbers, it follows that the Distance from 1 on A opposite b on B, shall be express'd by L : b—L : a, whence *per Prop.* 2. *Sect.* 2. the Number belonging to that Distance is $\frac{b}{a}=q$. Q. E. O.

Examp. Let the Divisor be 6, and dividend 96. According to the Rule I set 1 on A to 6 on B, then opposite 96 on B is 16 on A, which is the Quotient sought.

But here as well as in the preceeding Propositions, some seeming Difficulties analogous to those there mentioned may occur, but the same Remedy is sufficient. For let both the Terms be divided by any Power of 10, as 10, 100, 1000, &c. and the Quotient thence arising is to be multiplied by a Fraction, whose Numerator is the Divisor of the Dividend, and Denominator that of the Divisor; but both here and in the preceeding Proposition, 'tis scarce ever necessary that any more than one

one of the Factors, or Terms, need be divided; the Reason of this is evident, for since $\frac{b}{a}=q$; if m and n are any Powers of 10, then $\frac{a}{n})\frac{b}{m}(=\frac{nb}{ma}$, therefore $\frac{na}{mb}\times\frac{m}{n}=\frac{a}{d}=q$.

One Example will be sufficient to illustrate this: Let it be required to find the Quotient arising from 1944, being divided by 54; then calling the first 1 on *A one*, and the first 1 on B *ten*, and setting one on A to 54 on B, the other Term 1944 is not on B, but dividing by 10, the Quotient 194,4 on B is against 3,6 on A; this multipled by $\frac{10}{1}$ (for the above Reason) and I find 36 for the true Quotient.

The Rule of Three by the Sliding-Rule.

Prop. III.

Any three Numbers being given, 'tis required to find a fourth, which shall be to the third, as the second is to the first.

Solution. Set the first on B, to the second on A, then opposite the third on B, is the fourth on A.

For let us denote the three Numbers by a, b, c, and that sought by x, then $a:b::c:x$; therefore (*per* 16. VI*th* Euclid) $x=\frac{b\times c}{a}$, and (*per Prop.* 1. *and* 3 *Sect.* 2.) $L:x=L:b-L:a+L:c$.

Now when a on B is set to b on A, then 1 on B is distant from 1 on A by the Difference of the Spaces agreeing to a, b, that is by the Difference of their Logarithms; hence the 1 on B is distant from the same on A, by a Space expressed by $L:b-L:a$: and therefore the third Term on B is distant from 1 on A, by its Log. added to the former

former diſtance, *viz.* by $L:b-L:a+L:c$; but $L:b-L:a+L:c=L:x$, hence the Number on A oppoſite the third Term on B is $x=\frac{b\times c}{a}$. Q. E. O.

Here if when the firſt on B is put to the ſecond on A, the third cannot be found on B, or that being found, it happens beyond the Line A; any, or all the three Terms may be divided by any Power of ten, till all the four Terms happen on the Rule: then the Term thence ariſing is to be multiplied by a Fraction, whoſe Numerator is the Product of the Diviſors of the ſecond and third Terms, and Denominator the Diviſor of the firſt Term; for put m, n, r, to denote 10, 100, 1000, *&c.* the reſpective Diviſors of the firſt, ſecond, and third Terms, then $\frac{a}{m}:\frac{b}{n}::\frac{c}{r}:\frac{b\times c\times m}{a\times r\times n}$, but this multiplied by $\frac{n\times r}{m}$ gives $\frac{b\times c}{a}=x$, the ſame as was found before the ſaid Diviſion: And theſe are the Propoſitions whoſe Solutions are had by help of the Lines A, B, that is by two Lines of Numbers of equal Radii, or made from the ſame Scale of equal Parts. We now proceed to conſider thoſe which are had by a ſingle Radius, ſliding againſt a double or triple one, or ſuch where one is made from a Scale either double or triple to that from whence the firſt was made.

Of extracting the Square Root, by the Sliding-Rule.

PROP. IV.

Any Number being given, 'tis required to find its Square Root by the Sliding Rule.

The

The first 1 on *c* standing against the first 1 on D, against the given Number on *c* is its Root on D.

For since the Radius on D is double that on C, 'tis evident the Distance of 1 from any Number on D, is double the Distance of 1 and the same Number on C; therefore in this Position of the Rule, against any Number on D, is its Square on C, by *Cor. Prop.* 3. *Sect.* 2. whence on the contrary against any Number on *c* is its Square Root on D Q. E. D.

But if when the first 1's on each Rule are one, the Number given cannot be found on *c*, let it be divided by $\overline{10}|^2$ or $\overline{10}|^4$ or $\overline{10}|^6$ *&c.* till it can be found on *c*; then if the Number opposite this Quotient on D, is multiplied by the Square Root of the said Divisor, it will be the Root sought.

For let Q be the Number given, and *q* its Root, then $q^2 = Q$; but when Q is divided by any Number n^2, we have $\frac{Q}{n^2} = \frac{q^2}{n^2}$ and $\frac{q}{n} = \frac{\sqrt{Q}}{n^2}$, whence if this Root is multiplied by n, the Square Root of n^2 (the Divisor of Q), you'll have q, which by supposition is the Root of Q. *Q. E. O.*

Example. Let it be required to find the Square Root of 15129, the first 1's on C, D, being *one*, 'tis plain the greatest Number on C is 100, and on D 10, therefore I divide 15129 by $\overline{n}|^4 = 10000$, and the Quotient is 1,5129; against which on *c* is 1,23 on D, but because 15139 was divided by $\overline{n}|^4$, I multiply 123 by $\overline{10}|^2 = 100$, and that gives 123 for the Square Root of 15129.

PROP. V.

Any three Numbers being given, let it be required to find a fourth, which shall be to the third as the Square of the second is to the Square of the first.

Solution.

Solution. Set the first Root on D to the third Term on C, then on C opposite the second Root on D, is the Term required.

For let the three Terms given be denoted by a, b, c, and x the Term sought, then $a^2 : b^2 :: c : x$ by the Condition of the Question, or (by the 16th VI. *Euc.*) $x = \frac{b^2 \times c}{a^2}$, whence (from *Prop.* 1, 2, 3, of *Sect.* 2.) we have $L : x = L : 2L : b - 2L : a + L : c$.

Now when a on D is set to c on C, then b on D stands opposite a Number on C which is distant from 1, by the space belonging to the Number c, and the Difference betwixt the Spaces corresponding to b, a on D : but these Spaces being Logarithms of the Numbers themselves, and the Radius of D double that of C, 'tis evident the Difference of those Spaces on D is expressed by $2L : b - 2L : a$ to the Radius of C, to which the Log. of c being added, *viz.* $L : C$: we have $2L : b - 2L : c$, for the Distance of a Number from 1 on C to the Radius of C, opposite to b on D. Whence the Number on C agreeing this Distance is $\frac{b^2}{a^2} \times c = \frac{b^2 \times c}{a^2}$, but that by what was said above is equal to x, the Term sought. Q. E. O.

Example. Let the Numbers given be 4, 5, and 18, so that the Term or Number sought, is a fourth Proportional to $\overline{4}|^2 : \overline{5}|^2 :: 18$, by the Rule, I set 4 on D to 18 on C, then on C opposite 5 on D is $28\frac{1}{8}$, the Term or Number required.

But the chief Use of the Lines C, D, is in measuring of Cylinders, and consequently the Globe, Spheroid, Cone, and conic Conoids, which (excepting the Hyperbolics) are in a given Ratio to their circumscribing Cylinders. For it is demonstrated further on, if D be the Diame-

te:

ter of any Cylinder's Base, b its height, and $v=$ 1284, &c. the Diameter of a Circle whose Area is Unity, then the Measure of its Solidity is $\frac{D^2 \times b}{v^2}$ call it S, then $\frac{D^2 \times b}{v^2} = S$, or $v^2 : D^2 :: b : S$; whence from this Prop. we have the following Rule to measure any Cylinder by the Lines C. D.

Rule. *Set* 1,1284 *on* D *to the Cylinder's height* C; *then opposite the Diameter of its Base on* D, *is its Content on* C.

Let the Cylinder's Height be 6 Feet, and the Diameter of its Base 3 Feet 6 Inches.

Then setting 1,1284 on D, to 6 on C; against 3,5 on D is 57,7 on C: which is the Number of solid Feet in the Cylinder given. The same Difficulties may occur in the use of these Lines as in those above: for it may happen that after the first Root or Term on D, is set to the third Term on C, the third Term or Root cannot be found on D; or that being found, it happens off or beyond the Line C; in this case let the given Terms be each divided by 10, 100, 1000, &c. or any Multiple of these as occasion requires; then let the Term found be multiplied by a Fraction, whose Denominator is the Square of the Divisor of the first Term, and its Numerator the Product of the third Term, Divisor by the Square of that of the second, so shall you have the Number sought.

Example I. Let the three Numbers given be 3, 60, 7, so that the Term or Number required is a fourth Proportional to $\overline{3}|^2 : \overline{60}|^2 :: 7$, hence by the Rule I set 3 on D to 7 on C, then 60 on D is off the Rule, therefore I divided it by 10, and the Quotient 6 on D is opposite 28 on C; then because the first Term was divided by 10, the second by *one*, and the third by *one*, the multiplying Fraction

Fraction is $\frac{1\times 100}{\overline{1}|^2}=100$, hence $28\times1000=2800$ is the Term sought.

Example II. Let the three Terms given be 2, 80, 9, so that the Term sought is a fourth Proportional to $\overline{2}|^2 : \overline{80}|^2 :: 9 :$ —now setting 2 on D to 9 on *c*, then 80 is off the line D: let it be divided by 10, the Quotient 8 is on the Line D, but does not fall against any Division on C, whence divide 80 by 20, and the Quotient (4) on D, is against 36 on C, now here the multiplying Fraction is $\frac{1\times\overline{20}|^2}{12}=400$, therefore 14400 is the Number sought. But we may observe there is seldom occasion to divide any but the second Term, the Reason of this will be evident from hence; the three Terms given being *a*, *b*, *c* and *x*, that sought *x* is found equal to $\frac{b^2\times c}{a^2}$. Now if *a*, *b*, *c*, are divided by *m*, *n*, *r* respectively, then a fourth Proportional to $\frac{a^2}{m^2} : \frac{b^2}{n^2} :: \frac{c}{r}$ is $\frac{b^2\times c\times m^2}{a^2\times n^2\times r}$, whence this multiplied by $\frac{n^2\times r}{m^2}$ gives $\frac{b^2\times c}{a^2}$, which is the fourth Proportional to $a^2 : b^2 :: c$, the Term sought; but if the second Term *b* is only divided, the Term that is then found must be multiplied by n^2 only; because here *m* and *r* become each 1. We proceed in the next Place to shew the Use of the Line D E, in which the Radius of the former is triple the latter, it being laid down from a Scale three times as large as that of D.

The

The Extraction of the Cube Root by the Sliding-Rule.

PROP. VI.

Any Number being given, to find its Cube-Root by the Sliding-Rule.

The first 1's on the Lines D, E standing against each other, let the Number given be found on E, then opposite it on D is the Root sought. For the Radius of the line D being triple each one of E, every Number on the former is at a triple Distance from 1, that the same Number on the latter is from 1, hence the Number on D that stands opposite any Number on E, is the same that would be found at $\frac{1}{3}$ of the Distance on E, but these Distances are Logarithms of the Numbers: hence if Q be the Number given, and q its Root, then q^3=Q, or (*per Prop.* III. *Sect.* 2.) 3L : q=L : Q; therefore L : $q = \frac{L:Q}{3}$, that is $\frac{1}{3}$ part of the Distance between 1 and Q on E; to which corresponds the Number D opposite Q on E is the Log. of the Root sought, and therefore the Number it self is the Root of Q. *Q. E. O.*

But if Q is so large that it cannot be found on E, then let it be divided by $\overline{10}|^3$, or $\overline{10}|^6$, or $\overline{10}|^9$, and the Cube Root of this Quotient must be multiplied by the Cube Root of the said Divisor. One Example will be sufficient; Let it be required to find the Cube Root of 2197: Here the first 1 on E being *one*, the greatest Number on E is 1000, so that 2197 is not on the Line E, therefore I divide it by $\overline{10}|^3$, and on D against the Quotient 2,197, on E is 1,3; this is multiplied by 10, the Cube

Cube Root of $\overline{10}|^3$, gives 13 for the Cube Root of 2197. The Reason of this dividing, and its consequent multiplying may be shewn after this manner; Q is the Number whose Cube Root is q, let r be the Cube Root of $\frac{Q}{m^3}$, then $r^3 = \frac{Q}{m^3}$, or $r\,m = \sqrt[3]{Q} = q$, whence m being 10, 100, 1000, *&c.* or $m^3 = \overline{10}|^3$, $\overline{10}|^6$, 10^9, *&c.* the thing is manifest.

Prop. VII.

Any three Numbers being given, 'tis required to find a fourth, which may be to the third, as the Cube of the second is to the Cube of the first.

Solution. Set the first on D to the third on E, then opposite to the second on D is the fourth on E.

For let us again denote the three given Numbers by a, b, c, and x for that sought, then by the Condition of the Question $a^3 : b^3 :: c : x$, therefore by *Sect.* II. we have $L : x = 3L : b - 3L : a + L : c$. Now when a on D is set to c on E, then b on D stands opposite a Number on E, which is distant from 1 by the Difference of the Spaces 1, a : 1, b on D, and the Space 1, c, on E; therefore these Spaces being Logarithms of their Numbers, the Number on E opposite b on D is distant from 1 by a Space expressed by $3L : b - 3L : a + L : c$, to the Radius of E, but that by what was said above is equal to the Logarithm of x, hence the Number its self is equal to $\frac{b^3 \times c}{a^3}$, therefore the Proportional sought. *Q. E. O.* If either the first or second Terms cannot be found on D, they must be divided by 10, 100, 1000, *&c.* till they each obtain a Place thereon, and also till the Place of b (divided) on D, is opposite some

E division

division on E, and the Answer that thence arises must be multiplied by a Fraction, whose Numerator is the Cube of the second Term's Divisor, and Denominator the Cube of the first Term's Divisor; for the Terms being a, b, c, and the Term sought $\frac{b^3 \times c}{a^3}$ if for a, b, c, we put $\frac{a}{m}$, $\frac{b}{n}$, c, we shall have the fourth Term to these $\frac{b^3 \times c \times m^3}{a^3 \times n^3}$, whence if this be multiplid by $\frac{n^3}{m^3}$, it will give $\frac{b^3 \times c}{a^3}$ the Term required.

Examp. Let the Numbers given be 2, 80, 9, here when 2 on D is set to 9 on E, 80 is off the Line D, but dividing it by 10, against 8 (the Quotient) on D is 576; whence, because 80 was divided by 10, and 2 by 1, I multiply 576 by $\frac{\overline{10}|^3}{\overline{1}|^3} = 1000$, and it gives 576000 for the Term required.

To these three Proportionals we may add, that since it is demonstrated that all similar Planes are to each other as the Squares of their homologous Lines; and all similar Solids as the Cubes of their homologous Lines: thence by help of these three Lines C, D, E, all Spaces are measurable which are homologous to any given Plane; and all Solids which are Homologous to a given Solid.

We are now got to the Line MD, whose first Division at the right hand corresponds to ,215042, or 2,15042, or 21,5042, or 215,042, &c. by which means the first 1 thereon is against 2,15042, or 21,5042, or 215,042, &c. respectively on A; the Values of the Divisions on MD increasing towards the left hand, as those on the rest do towards the right. Why it should begin thus, and not at *one*, as the others do, will be evident after we

we have laid down the two following Propositions, but we must observe the use to which this Line with the others on this Side the Sliding-Rule are applied, is in *Malt-Gauging*; the Vessels in which that is usually put being in the Forms of Parallelopipedons, having Rectangular Bases. Now if the Length of any Parallelopipedon be l, its Breadth b, and Height or Depth h in Inches, $l\times b\times h$ is the Number of Cubick Inches it contains, as is demonstrated farther on; or the Area, or Measure of the Base being q, the Cubic Inches thereof is expressed by $q\times h$, but there are 2150,42 Cubic Inches in a Bushel of Malt, therefore $\frac{b\times l\times h}{2150{,}42}$, or $\frac{q\times h}{2150{,}42}$, is the Bushels of Malt in the said Vessel, whether Floor, Couch or Cistern, having the above Form.

Prop. VIII.

The Measure of the Sides of any rectangular Parallelogram being given, to find its Measure by the Lines B *and* MD.

Solution. Set the Length on B to the Breadth on MD, then opposite 1 (or unity) on MD is the Measure on B.

For the Measure of the Breadth being b, and that of its Length l, the Measure of the Rectangle is $b\times l$, as is demonstrated farther on, hence $q=b\times l$, or $L:q=L:b+L:l$: (*per Prop.* I. *Sect.* 2.) Now when b on B is set to l on MD, then 1 on MD is against a Number on B, which is distant from b towards the right hand by the Space 1, l, so that 1 on MD is against a Number on B, which is distant from 1 by the Sum of the Spaces 1, b, and 1, l, but those Spaces are Logarithms of their Numbers; whence the Distance of the Number

on B, opposite 1 on MD is expreſs'd by L : b + L : c; and therefore from what was ſaid above, the Number it ſelf is L$\times b$ or q. *Q. E. O.*

Of Meaſuring Floors, Couches, or Ciſterns by the Sliding-Rule.

Prop. IX.

The Length, Breadth, and Height (or Depth) of any Couch, Floor, or Ciſtern being given, to find the Number of Buſhels of Malt it contains.

Solution. Set the Length on B to the Breadth on MD,, then againſt the Height on A is the Content on B.

For the Length being l, Breadth b, Height h, and the Area of the Baſe $q=l\times b$; the Content is $\frac{q\times h}{2150,42}$ as obſerved before: but by the laſt *Propoſition* q is againſt 1 on MD, therefore by *Propoſition* 3, foregoing, if 2150,42 on A be ſet to q on B, then oppoſite h on A is $\frac{h\times q}{2150,42}$ on B. But we obſerved above, that 2150,42 on A, is oppoſite 1 on MD, and 1 on MD has been already ſhewn to be againſt q on B, therefore 2150,42 on A is againſt q on B, whence againſt h on A is the Content on B, *viz.* $\frac{h\times q}{215042}$. *Q. E. O.*

In the Uſe of theſe Lines, A, B, and MD together, there may ariſe ſome Difficulty about eſtimating the Value of the Numbers found by theſe Directions, which muſt be cleared up before we finiſh this Chapter of the Uſe of the Sliding-Rule.

In the firſt place therefore it is to be noted, you muſt always have 1 on MD, repreſenting Unity; other-

otherways the Area of the Base or Product of $l \times b$ cannot be directly found on B : but you'll have $\frac{1}{100}$, $\frac{1}{10}$, ten times, 100 times, *&c.* that Area according to the Value put for the first 1, to the right hand MD ; wherefore that the direct Value may be had, let that ever stand for $\frac{1}{10}$ or 1. If it represent 1, then the Number opposite on A (where is a Brass Pin marked MB) should be 2150,42, the cubic Inches in a Bushel of Malt; but by having that so large, it will commonly happen that the Height or Depth which is to be found thereon, is off the Line, for the least Number on that Supposition is 100; wherefore to remedy that, you are to diminish the several Measures by dividing them by 10, 100, 1000, *&c.* till each (when the first or second 1 on MD is unity) may not only have a place on its respective Line, but that the Term required may also have its place on B: and whatever is the Product of all these Divisors of *l*, *b*, *h*; the Number 2150,42 must be divided thereby, by which the Value of the Divisors of A will be known. The Reason of this is thus made out, put B the Bushels of Malt in the Cistern, Couch, *&c.* whose Length is *l*, breadth *b*, and height *h*, then $B = \frac{l \times b \times h}{2150,42}$: But let each of these Measures be divided by *m*, *n*, *r*, respectively, which are always 10, 100, 1000, *&c.* then I say, if 2150,42 is divided by the Product of these Divisors, the Product of the former Quotient divided by this latter is still equal to B, the Content of the said Couch or Cistern, *&c.*

for $\frac{2150,42}{m \times n \times r}\Big) \frac{l}{m} \times \frac{b}{n} \times \frac{h}{r} \Big(\frac{l \times b \times h}{2150,42} = B.$

We shall make this Observation plain by Examples.

Example 1. Let the Breadth of a Couch be 56,2 Inches, its Length 270, and Height 5,2 Inches;

now in order that 5,2 may be found on A, the first 1 must be either $\frac{1}{10}$ or 1; let it be 1, then the second MB on A is 21,5042; whence in this case 2150,42 is divided by 100, therefore let 270 be divided by 10, and 56,2 by 10, then setting 27 on B to 5,62 on MD, against 5,2 on A, is 36,7 on B, which is the Number of Bushels in the Couch proposed.

Example 2. Let the Breadth be 72 Inches, Length 140, and Depth 18,2. Now that 18,2 may be on A, the first 1 may be *one* by which the second MB is 21,5042, so that 215042 is again divided by 100, therefore let 140 be divided by 10, and 72 by 10; then setting $14=\frac{140}{10}$ on B, to $7.2=\frac{72}{10}$ on MD against 18.2 on A, is 85,3 on B, which is the Content sought.

Example 3. Let the Breadth be 13 Inches, the Length 36 Inches, and the Depth 9 Inches, therefore the first 1 on A may be either $\frac{1}{10}$ or 1, for in both cases 9 may be found on A; let it be 1, then (the second Brass Pin being mark'd MB) is 21,5042, so that 2150,42 is divided by 100; therefore let the Breadth be divided by 10, and the Length by 10, so that both together are divided $10 \times 10 = 100$, and they become 1.3, and 3.6; then setting 3.6 on B, to 1.3 on MD against 9 on A, is 1.96 on B for the Content.

Example 4. Let the Length of a Floor be 1250 Inches, Breadth 360 Inches, and Depth 9 Inches, and let the first 1 on A be $\frac{1}{10}$, then the second MB is 2,15042, so that 2150,42 is divided by 1000; therefore let 1250 be divided by 10, and 360 by 100, so we have 125 and 3.6; then setting 125 on B to 3.6 on MD against 9 on A, is 1.884 on B, which is the Content sought.

So that you see the first 1 to the right hand on MD has ever been *one*, the first *one* on B either 10, or 100, or 1000, &c. and the first 1 on A was varied

varied as occasion required, which was always determined so as the Height might be found thereon, and also that that might fall against some Number on B; but from thence we determined by what the Divisor 2150,42 was divided, and then the Divisors of the Length and Breadth: For the Product of their Division must ever be equal to the Divisor of 2150,42. In the last Case, the second Point MD being 2150,42 was taken for 2,15042; so that it was divided by 1000, and the Length 1250 was divided by 10, and the Breadth by 100, because 10x10=1000, the Divisor of 2150,42; so that from these Directions the Reader will never be at a loss to estimate the Value of the Divisors on the Sliding-Rule according as occasion presents.

The frequent use that is made of this Rule by my Brethren of the Excise, made me the fuller in this particular, that they might not only be acquainted with its Use; but also the Reason of the Method of Operation, which I have laid down in a manner as strictly mathematical, as the Nature of the Thing admits, and yet so illustrated by verbal Rules and Examples, that I would fain hope it will not be out of the reach even of the Learner, which I think has not been done before; so that we shall dismiss it for the present till we come to the Practice, where the Reader will find a further Application of it in measuring the various Solids that occur in GAUGING.

PART II.

CHAP. I.

Of Geometrical Definitions and Theorems.

IN the preceding Chapters we have laid down the Method of computing, both by the Pen and Sliding-Rule; it remains that we ſhew how thoſe Rules are applicable in finding the Contents or Meaſures of Magnitude. For as on the one hand, knowing for Example, that the Meaſure of any Parallelopipedon is equal to the Product of the Meaſures of its Baſe and Height, I ſay as this could be of little Service, without knowing how to compute by ſuch Meaſures: So on the other hand it would be of little Uſe to be able to compute without knowing when to multiply, when to divide, &c. and alſo the Terms by which thoſe Operations are to be made. Therefore having laid down the Method of computing, we ſhall now proceed to ſhew how thoſe Rules are to be made uſe of in Meaſuring of Magnitude.

Definitions.

1. *Magnitude* is whatever is extended, and therefore may be conceived to be contained or limited by ſome certain Term or Terms.

1. A *Point* is that which has no Parts, or conſidered as indiviſible, and ſo of it ſelf no Magnitude, and yet in ſome meaſure may be taken for the beginning thereof. For,

3. If we conceive A (*Fig.* 6.) to be a Point, and any how moved towards B, by this Motion we may conceive to be generated a Magnitude AB, of one Dimen-

Dimenſion only, that is, Length without Breadth, or Thickneſs; and this is called a *Line*.

4. When A moves towards B the ſhorteſt way, AB is called a *Right Line*.

5. But if the Direction of the point A be changed in every point of its Path or Way, as A*c*, *bc*, *be*, *&c.* (*Fig.* 7.) then the generated Magnitude is a *Curved Line*; the Species of which are infinite; but ſuch of them as are uſeful in Gauging, ſhall be conſidered farther on.

6. If the Line AB be alſo conceived to move in a rectilineal Direction E *e*, its extreme Points A, B will deſcribe other Lines AC, AB, and the Line itſelf will have generated a Magnitude AD of two Dimenſions, having both *Length* and *Breadth*, but no Thickneſs; and this is called *a Plane Surface*, or ſimply a *Plane*.

7. If this Surface AD be conceived to move forward or backwards in the rectilineal Direction E *e* (*Fig.* 8.) its oppoſite Points A, D, B, C, will again deſcribe other Lines AF, DK, BH, CG, *&c.* and conſequently each of theſe Lines other Planes, and thence by this Motion we may conceive to be generated a Magnitude of three Dimenſions, *Length*, *Breadth*, and *Thickneſs*, called a *Solid*. But here as before in the Generation of Lines, ſhould the Direction of the Generating Line, or Plane, be a curve Line, then the Plane, or Solid, will accordingly be contained by a curve Line, or curved Solid.

We gave this Generation of Magnitude, becauſe of its great Uſe and Extent in the Doctrine of curve Lines, and conſequently the whole of Mathematical Philoſophy. But yet I. muſt confeſs, by thoſe of a wrangling or ſophiſtical Temper, it may be objected that we have given the Generation of a Subſtance, from a Nothing, or an Attribute, which is a ſort of Contradiction; for it's moſt certain, Points and Lines

are

are only Attributes of Body. And some whose Geometry seems to be confined to seeing only, *have objected to the Elementor, for his negative Definitions of a Point, Line and Surface; or rather denied that such Points, Lines and Surfaces exist at all. To the first it may be replied, We do not so much pretend to* generate Magnitude, *that would be absurd indeed; but only to convey to the Mind the Idea of the Figure which the several Sorts of Magnitude really in being may put on. And to the second we must observe, Method obliged the Elementor, whom we have imitated in this, to begin with the most simple first, and therefore he began with defining a Point, where he only tells us what it is not. They will reply, of nothing or non-entity there are no Properties or Affections: But we answer, Points or Lines are the Attributes of Body, and where that exists, they must also; for by first assuming some Body to be given, it is certain it has Ends; then if those Ends are flat, or exactly even, they will be* Planes, *and those Planes have also their Ends, which if streight, they are* Right Lines, *and those Lines have Ends, which are Points. So that from hence 'tis plain we have adequate Ideas of Points, Lines, and Plains, as defined by the Geometers; those Points and Lines drawn with a Pen, being only Pictures of the real Points or Lines which Mathematicians consider, and drawn to assist the Imagination.* But to proceed;

8. A *Circle* is a plain Surface, terminated by one curved Line called *Circumference*, unto which all Lines, drawn from a Point within called the *Center*, are equal to each other; these Lines are called *Semidiameters*, or *Radii*; and any Line passing thro' the Center, and terminated by the Circumference, is called a *Diameter*; the Line joining the Ends of any part of the Circumference, is called its *Chord*.

9. A *Sector* of a Circle is part thereof, terminated by two Semidiameters, and part of the Circumference, either more or leſs than the half; and this Arch is called the *Baſe* of the Sector.

10. A *Segment* of a Circle is part thereof, terminated by part of the Circumference and its Chord; and when that part is half of the Circumference, it is called a *Semicircle*.

The Ancients divided the Circumference of a Circle into 360 equal Parts, each of which they called a *Degree*; and each Degree into 60 equal Parts called Minutes; and each Minute into 60 Parts called *Seconds*, and ſo on to thirds, fourths, *&c.* every Arch of a Circle, being denoted by the Number of Degrees, Minutes, Seconds, *&c.* it contains.

11. If two Lines in the ſame Plane, and not lying in the ſame Direction, are ſo ſituate, that being produced they would meet, they are ſaid to be *inclined to each other*; and if two inclined Lines are produced till they do meet, they will form a *Plane Angle*; The Point of Concourſe is called the *Angular Point*, and the Angle is ſaid to be *more* or *leſs*, according as the Inclination of its Lines are *more* or *leſs*; but the Quantity thereof is eſtimated by the Arch of a Circle contained betwixt its Lines, and having the angular Point for its Center. Every Angle may be denoted by three Letters, the middlemoſt of which is at the Angular Point, and the other two at the Ends of the Lines. Thus (*Fig.* 9.) the Angle form'd by the Lines BD, CE produced, is denoted by BAC.

12. When a ſtraight Line ſtands on another, ſo that it does not incline to one end more than the other, but makes the Angles on both ſides equal, then are theſe called *Right Angles*; and the Lines are ſaid to be perpendicular to each other.

Hence

Hence the Measure of the Right Angle is 90 *Degrees.*

13. Every Angle *greater* than a Right one, is called an *Obtuse Angle*; and every Angle less than a Right one, is called an *Acute Angle*; or in general both obtuse and acute are called *Oblique Angles.* Thus,

$$\text{Figure 10.}\left\{\begin{array}{l}\text{CBD}\\\text{CBE}\\\text{ABD}\end{array}\right\}\text{is an}\left\{\begin{array}{l}\text{Acute}\\\text{Right}\\\text{Obtuse}\end{array}\right\}\text{Angle.}$$

And the Lines AC, EB are perpendicular to each other.

14. If to one end of any Arch of a Circle a Diameter be drawn, and from the other end a Perpendicular be let fall thereon, it is called the *Sign* of the said Arch. And if from that end of the said Diameter, which co-incides with the Arch, a Perpendicular be erected, and a Line drawn from the Center of a Circle and the other end of the Arch, and produced till it meet the same; then that part thereof which is betwixt the Point of Concourse and Center, is called the *Secant* of the Arch; and the part of the Perpendicular betwixt the Point of Concourse and Diameter is called the *Tangent* of the Arch. Also that part of the Diameter betwixt the Sine and Tangent of any Arch, is called its *Versed Sine*: Thus in the Circle ACBA, (*Fig.* 11.)

$$\left.\begin{array}{l}\left.\begin{array}{l}\text{CEB}\\\text{CEA}\end{array}\right\}\\\left.\begin{array}{l}\text{BHED}\\\text{AKEA}\end{array}\right\}\end{array}\right\}\text{are}\left\{\begin{array}{l}\text{Sectors.}\\\\\text{Segments.}\end{array}\right.$$

ABIA is a Semicircle.

$$\left.\begin{array}{l}\text{ES}\\\text{BT}\\\text{CT}\\\left.\begin{array}{l}\text{BS}\\\text{AS}\end{array}\right\}\end{array}\right\}\text{is the}\left\{\begin{array}{l}\text{Sine}\\\text{Tangent}\\\text{Secant}\\\text{Versed Sine}\end{array}\right\}\text{of the Arch EB.}$$

15.

15. Those Right Lines drawn on the same Plane are said to be parallel, which being produced ever so far would not meet, or the Parallels are such, that if from ever so many Points in the one, Perpendiculars be let fall on the other, they will be all equal to one another. Thus, if AB and CD, (*Fig.* 12.) are two Lines so situated to one another, that from the Points *a*, *a*, *a*, in AB, the Lines *ab*, *ba*, *ab*, being let fall perpendicular on C, the several Lines *ab*, *ab*, *ab*, are equal among themselves, then is AB parallel to CD.

Of Right-Lined Surfaces, or Planes.

16. *Right-Lined Surfaces* are those Planes which are included by Right Lines, and the including Lines are called their Sides.

17. A *Triangle* therefore is a Plane included by three Right Lines, of which respecting its Sides, that is *equilateral*, whose three Sides are equal, (*Fig. a.*) that an *isosceles Triangle* having two Sides equal, (*Fig. b.*) and a *scalene*, or *oblique Triangle*. when all the Sides are unequal. (*Fig.* 16.) But respecting its Angles, that is a *right-angled Triangle* which has a right Angle, (*Fig. c.*) and an acute-angled Triangle when all the Angles are acute, (*Fig. a.*) and an obtuse *angled Triangle*, when it has one obtuse Angle. (*Fig.* 16.)

18. In right-angled Triangles, the two Sides containing the right Angle are called its Legs, and the Side subtending the right Angle is called the *Hypothenuse*.

19. If a Perpendicular be let fall from any Angle of a Triangle to its opposite Side, that Side is then called the *Base* of the Triangle; the Perpendicular its *Height*, and the Point from whence the Perpendicular was drawn, is called its *Vertex*.

20. Of *four-sided Figures*, those are called *Parallelograms*, whose opposite Sides are parallel or equal;

equal; and these are distinguished into four Sorts; 1. When the Sides are all equal, and Angles right, they are called *Squares*, (*Fig.* 13.) 2. When the opposite Sides are only equal, and Angles Right, *Rectangles*, (*Fig.* 14.) 3. When the Sides are equal, and Angles oblique, *Rhombus's*, (*Fig.* 17.) 4. When the opposite Sides are only equal, and Angles oblique *Rhomboids*, (*Fig.* 15.) All other quadrilateral Figures called *Trapezias*, (*Fig.* 18.)

Of multilateral or polygonal Figures.

21. A *Pentagon* is contained by five equal Sides, (*Fig.* 20.) A *Hexagon* by six. A *Heptagon* by seven. An *Octagon* by eight, *&c.* Each taking its Name from the Number of equal Sides or Angles by which it is contained.

To this Definition we may add, that all Polygons may be circumscribed by a Circle, or have a Circle inscribed in them: For if any two contiguous Sides of a Polygon are perpendicularly bissected, the cutting Lines will meet in a Point, which is the Center both of the circumscribed and inscribed Circles; the Distance of that Section, and any Angle of the Polygon, being the Radius of the former, as the Distance of the said Section, and middle of any side is of the latter: Whence the Center of the inscribed or circumscribing Circle of any Polygon, is called the Center thereof; and that not improperly, it being its Center of Gravity. Thus (*Fig.* 20.) DC and EC being drawn perpendicularly thro' DE, the middle of AB, BE, intersect each other in C, the Center of the Polygon; whence CB is the Radius of the circumscribed, and CD=CE of the inscribed Circle.

22. All multilateral Figures of unequal Sides, are called *irregular Polygons.*

23. The Length of a Perpendicular drawn from any Angle of a Parallelogram to its opposite Side,

is

is called its *Height*; and the Side on which it falls is then called its *Base*.

We gave these elementary Definitions for the sake of those unskill'd in geometrical Subjects; but when we have occasion, in what follows, to mention any Properties of the things defin'd above, we must refer the Reader to *Euclid*; for our intended Brevity would not permit giving a regular Method of Demonstration, and therefore we chuse to omit them altogether.

Of the Measure of Magnitude.

23. Every Magnitude is measured by another of the same kind: As a Line is measured by a Line, a Surface (or Plane) by a Plane, and a Solid by a Solid: And the same extends to Velocity and Tardity in Motion, Intention, and Remission in Qualities, and so on, of every thing to which the Words *more* or *less* can be apply'd.

24. And that Magnitude by which another is measured, we call its *measuring Unit.* This is arbitrary, and may be assigned by the contracting Parties at pleasure. *The measuring Unit for Lines may be a Line of any Length*; but for short Lines is commonly a Foot, or twelve Inches. For *Surfaces*, the measuring Unit is a Square, whose Side is the *measuring Units for Lines.* And the measuring Unit for Solids is a Cube, *whose Side is equal to the measuring Unit for Surfaces.*

25. The Measure of any Magnitude is the *Ratio* it has to the measuring Unit, or may be said to be the Number of Times and Parts of a Time that it contains the measuring Unit. Thus if the Line *ab* (*Fig.* 13.) be taken for the measuring Unit, and the Line AB be four times as long, then its Measure is 4; for AB : *ab* : : 4 : 1, or $\frac{AB}{ab}=\frac{4}{1}=4$.

Again, if *a b c d* (*Fig.* 14.) be a Square, whose Side is equal to *ab* in the last, and ABCD is a Space

Space 12 times as much; then the Meaſure of ABCD is 12; for ABCD : *abcd* : : 12 : 1, or $\frac{ABCD}{abcd}$ $=\frac{12}{1}=12$. *So that from hence we may ſee by the Meaſure of any Magnitude, is meant its Numeral Relation to another of the ſame kind eſteemed as Unity.*

Of meaſuring right-lined Surfaces.

Prop. I.

The Meaſure of every Rectangle, is equal to the Product of the Meaſures of its containing Sides.

Let the Meaſure of BD, one of the Sides of the Rectangle ABCD (*Fig.* 14.) be called l; and the Meaſure of AB, (the other containing Side) b; then if B*a*, *ag*, *gb*, &c. are taken each equal to the linear Unit, their Number will be l: through each of which draw Lines parallel to BD, meeting DC in as many Points *r*, *s*, *t*, &c. then if B*d* and *a c* are each taken equal to B*a*, 'tis evident (from *Definit.* 24.) B*acd* will be the ſuperficial Unit; and let *d e*, *e f*, &c. be taken each equal to B*d*: Now 'tis manifeſt the Meaſure of B*ar*D will be $l\times 1$, for B*ar*D : B*acd*: : $l\times 1$: 1; whence (*per Definit.* 25.) $l\times 1$ is the Meaſure of B*ar*D; but the Spaces B*r*, *as*, *g t*, &c. are equal (by 36. 1. *Eucl.*) therefore any one of them ſo often repeated as is their Number, will be the Meaſure of the whole Figure, but b is put for that Number, therefore $b\times l\times 1=bl$, *viz.* the Product of the Meaſures of the contained Sides, is the Meaſuree of the Rectangle ABCD. *Q.E.D.* But if betwixt the Points B*a*, any Point *v* was taken, and the Line *v w* drawn parallel to BD, ſo that a rectangular Space BD*wv* was to be meaſured, whereof one ſide was leſs than

than the linear Unit, the Measure of that Space will be expressed as a Fraction, whereof the Measure of the Side BD is Numerator, and that of $\frac{1}{Bv}$ Denominator; for let $\frac{1}{p}$ express what part B*v* is of B*a*, or Unity; then the Measure of B*v* is $\frac{1}{p}$, and (from the 1. 6th *Euclid*) BD*wv* is the same Part of BD*ra* that B*v* is of B*a*, but that is $\frac{1}{p}$ by supposition; and from what was said above, the Measure of BD*ra* is $l \times 1$, or l; therefore $BvwD = l \times \frac{1}{p}$, *viz.* equal to the Product of the Measures of the containing Sides, whence the Proposition is true universally.

Scholium. Multiplication, properly speaking, can only obtain where the Multiplier is a whole abstract Number; but for want of a better Word, it is commonly used tho' that was a Fraction, or Surd, because thence is produced a Number, which has the same Ratio to one Factor, that the other has to Unity; which is also the case when they are whole Numbers; for the Factors being f, F, and the Product p, we have $f \times F = p$, or $p : f :: F : 1$.

Cor. 1. *Hence the Measure of every Parallelogram is equal to the Product of the Measure of one of its Sides, into a Perpendicular let fall thereon from the other.*

Thus (*Fig.* 15.) put p for the Measure of the Perpendicular AP, and s for the Measure of the Side CD=AB, then $p \times s$ is the Measure of ABCD; for if CD be continued to Q, where it meets a Line drawn thro' B parallel to AP, then (by the 35 *Euclid.* 1.) APQB is equal to the Parallelogram ACDB; but the Measure of the former is $p \times s$, and therefore $p \times s$ is also that of the latter: *Whence to measure any parallelogramick Space by the*

Sliding-Rule, we have this Rule: Set 1 on B to the Length on A, then opposite to the Breadth on B, is the Content on A, (by *Prop.* 1. *Sect.* 2.)

Example. Let AB=12.55 Inches, and AC=8.32: then the Measure of the ABCD is $12.55 \times 8.32 =$ 104,416 square Inches. Or, *by the Sliding Rule*, 1 on B set to 12.55 on A against 8.32 on B, is 104.416 on A, which is the Measure sought.

Cor. 2. *Hence the Measure of every right-lined Triangle is equal to half the Product of the Measure of one of its Sides into the Measure of a Perpendicular drawn thereon, from its opposite Angle.*

Let the Measures of the Sides CD, DB, BC, (*Fig.* 16.) of the Triangle BCD be a, b, c, respectively, and the Measures of the Perpendiculars BQ, CP, DR, p, q, r, respectively, then the Measure of BCD is $\frac{a \times p}{2}$, or $\frac{b \times q}{2}$, or $\frac{c \times r}{2}$; for thro' B draw a Line parallel to CD, and another thro' C, parallel to BD, meeting the former in A, then the Parallelogram ABCD is double the Triangle BCD (*per* 42 *Euclid.* 1.) but the Measure of ABCD by the preceding *Corol.* is $a \times p$, therefore $\frac{a \times p}{2}$ is the Measure of the latter. The other two Expressions may be proved after the same manner.

Cor. 3. From hence we deduce the manner of measuring all right-lined Spaces universally; which is done by dividing the Figure into the least Number of Triangles possible, these will ever be *less by two* than the Number of the Figure's sides; and the Measure of each of them being found by the last *Corol.* their Sum will be the Measure of the Figure proposed. But that may be effected at one Operation, by first reducing the given Figure into a Triangle; the Method for doing which will be manifest from this

PROBLEM.

PROBLEM.

To reduce any Trapezia, or Figure, contained by four right Lines into a Triangle.

Let the four-sided Figure be ABCD, join any two opposite Angles as B, D, by the Line BD; also thro' C, or A, the other Angles, draw a Line parallel to the Diagonal Line BD; let it meet the Sides AD or CB produced in E or F, and draw the Line BE or DF, the first of which cuts CD in K, and the second AB in L, then shall the Triangles BAE and CDF be each equal to ABCD: for the Triangle CDE is equal to CBE, (*per* 36 *Eucl.* 1.) therefore the Triangle DKE is equal to BCK, to which BADK being added, we have BAE equal to ABCD; and after the same manner CDF is proved equal to ABCD. *Q. E. F.*

Therefore if the Figure has five Sides, first divide it into a four-sided Figure and a Triangle, by drawing a Line from any Angle to the next but one: the former of which is reduced to a Triangle by the above Problem, and this Triangle, with what the said four-sided Figure wants of that given, will form another four-sided Figure; which being reduced by the above Problem, that will be the Triangle sought: And after the same manner you may proceed if it has six, seven, eight, or ever so many sides, which the Reader will more readily perceive by consulting the 18 and 19 *Figures*; the first ABCDE is a five-sided Figure, and the second ABCDEF has six, which are reduced into the Triangles AFG, GHK.

Cor. 4. *From* Cor. 2. *we also infer, that the Measure of every Polygon, is equal to the Sum of the Measure of its Sides into the Measure of a Perpendicular thereon from the Center.*

But since the Perpendicular from the Center of a Polygon is required to find its Area, we shall

lay down a Method by which the Area of any Polygon may be directly found from its Side being given: In order to which, let n denote the Number of Sides of the Polygon, and from the Table of natural Signs take the Sine and Cosine of $\frac{180}{n}$ Degrees; then as four times the Sign of $\frac{180}{n}$ Degrees, is to the Cosine thereof, multiply'd by n, so is the Square of any Polygon's Side, to its Area: For the Product of the Sine and Cosine of $\frac{360}{2n}$, or $\frac{180}{n}$ Degrees, is the Area of a Triangle in a Polygon, the Radius of whose circumscribed Circle is 1, similar to any other Polygon of n Sides; but like Triangles are as the Squares of their homologous Sides, and twice the Sine of $\frac{180}{n}$ Degrees is the Base of the Polygon, the Radius of whose circumscribed Circle is Unity. Hence if the Sine and Co-sine of $\frac{180}{n}$ are called s, c, respectively, also S the Side, and A the Area of the Polygon given, then $4s^2 : s \times c \times n :: S^2 : A$. or $4s : cn :: S^2 : A$.

From whence we deduce this Rule; *To find the Area of any Polygon having its Side given*. Let 180 Degrees be divided by the Number of the Polygon's Sides; and let n be divided by four times the Tangent of that Quotient (taken from the Table of natural Tangents) then if the Quotient of that Division is multiply'd by the Square of the Side of the Polygon given, the Product will be the Measure of its Area.

Example. Let the Side AB (*Fig.* 20.) of a Pentagon be 50 Inches, and its Area or Measure required; here, according to the Rule $\frac{180}{5} = 36$, whose

whose Tangent from the natural Tables is ,7265435 by four Times, which (*viz.* 2,906178) $n=5$ being divided, we have 1.720477 for the Quotient; and this multiplied by 2500, the Square of the Side (50) it gives 4301,175 square Inches for the Area. But because it would be troublesome always to have Recourse to the Table of natural Tangents, I have thence selected the Tangents of $\frac{180}{3}=60$, $\frac{180}{4}=45$, $\frac{180}{5}=36$, $\frac{180}{6}=30$, *&c.* Degrees, which answer to half the Angles at the Center of a Polygon of 3, 4, 5, 6, 7, *&c.* Sides; as also the Quotients arising from dividing n by the Quadruples thereof, as in the following Table.

A Table for measuring POLYGONS.

The Names of the Polygons.	N° of Sides.	Half the Angle at the Center.	The Natural Tangent of the last Column.	Second Column divided by Quadruple the 4th.
Equilat. Triangle	3	60° 00′	1.7325058	.433013
Square	4	45 00	1.0000000	1.000000
Pentagon	5	36 00	.7265425	1.720477
Hexagon	6	30 00	.5773502	2.598076
Heptagon	7	25 42 6/7	.4815712	3.633959
Octagon	8	22 00	.4040262	4.828427
Nonagon	9	20 00	.3639702	6.181824
Decagon	10	18 00	.3249196	7.694209

 This

This Table may be continued at pleasure, by the Rule laid down for that purpose, but from this only the Area of any regular Polygon of less than eleven Sides is had directly by a simple Multiplication, as we explained before. But from hence we also may give the Method of Measuring any of those Polygons by the Lines C, D, on the Sliding-Rule; for by calling any Term in the fifth Column q, and the Side of the Polygon S, by this Proportion the Area is found, *viz.* $1 \times 1 : q :: S^2 :$ (a) the Area: therefore (by *Prop.* 5. *Sect.* 2. *Part* 1.) we have this Rule: *Set* 1 *on the Line* D *to* (q) *any Number in the fifth Column on* C, *then on* C *opposite the Side of the Polygon given on* D, *is the Content or Area sought.*

Thus in the last Example, the Side of a Pentagon was 50 Inches, therefore because the Number in the fifth Column opposite Pentagon is 1.720477, I set 1 on D to 1.72 on C, then opposite 5 on D (for 50 is of the Rule) is 43.01 on C, and because 50 was divided by 10, I multiply 43,01 by 100, (the Square of 10) and find 4301 for Area sought. See *Prop.* 5. *Sect.* 2. *Part* 1.

Having now shewn how to find the Measures of Plains contain'd by right Lines only, we should proceed to those contained either wholly by a curve Line, or partly by a curve Line, and partly by a straight one; but first we must lay down the following Properties of the Conic Sections, because every Reader whose Business requires Skill in Gauging, cannot be suppósed furnish'd with a Treatise of that kind, they not being so commonly read as the *Elements of Euclid*.

CHAP. II.

Of the Elements and some of the essential Properties of the Conic Sections.

Defin. 1. IF (AC Fig. 21, 22, 23, 24, 25,) one Leg of a right-angled Triangle ABC remains fixt, and the whole Triangle be conceived to turn thereon as an Axis, till it return to the Place from whence it moved, it will generate the likeness of a Solid called a *right Cone*; of which the fixt Side AC is called the *Axis*, the other Leg BC describes a Circle called the *Cone's Base*; and AB the Hypothenuse will describe the *Conic Surface*. A, the Vertex of the generating Triangle, being also the Vertex of the generated Cone.

2. If the Hypothenuse AB should pass the Vertex A, the external Part AB will generate another similar Cone A*bd*, and both together are called *opposite Cones*. See *Fig.* 25.

Cor. 1. Hence every Plane drawn thro' the Vertex and Center of the Base is perpendicular thereto, (18 *Euclid* 11.) for it passes thro' the Axis AC, which by Supposition is perpendicular to the Base of the Cone.

Cor. 2. *Hence the common Section of every Plane drawn thro' the vertex, with the Cone, is a right-lined Triangle.* For let ABMDN (*Fig.* 1.) be the Cone, and AMN the Section made by a Plane passing the Vertex A; also let C be the Center of the Cone's Base, draw the Lines CM, and CN, then 'tis evident the two right-angled Triangles ACM and ACN are the Positions of the generating Triangle, when the Point B is in M, N, respectively; therefore their Sections with the Plane AMN, are two Right AM, AN, (8 *Euclid.* 11.) which are also the

the Sections of the said Plane and Cone; and MN the Section of the said Plane and Base, is also a right Line, wherefore AMN the Section of the Cone by a Plane thro' its Vertex, is a right-lined Triangle.

Cor. 3. *Hence also the Section of the Cone, by a Plane Parallel to its Base, will be a Circle.* For (*Fig.* 22.) let *o* be any Point in the Axis of the Cone, thro' which draw a Line *o* F, parallel to the Base of the Triangle, CB, then 'tis manifest while the Side BC describes the Circle BMDN, which is the Cone's Base, *o* F will describe another Plane similar to it, but *o* F in every Point of its Path retains its Parallelism to BC; wherefore in its Motion it will describe the same Figure that would be made by a Plane passing thro' *o*, parallel to the Base of the Cone; but the former has been shewn to be a Circle, therefore so is the latter.

Defin. 3. If a Cone be any how cut by two Planes, one thro' its Vertex and Center of the Base making the Triangle ABD, (*Fig.* 23.) and another perpendicular thereto, and so as it cuts both the Sides of the Triangle ABD, but oblique to the Base BD, the Figure of the Section is a curve Line called an *Ellipse*; if V*v*, denotes the Line where the elliptic Plane cuts the Triangle ABD, then V *v*, is called the *transverse Axis* of the Ellipse; and V, *v*, its *Vertexes*.

Defin. 4. The Parts of Lines drawn within the Ellipse at right Angles to the Transverse, and intercepted by the same and Curve, are called *Ordinates* to the Axis. The Middle of the Transverse is the Center of the Ellipse, and that double Ordinate which passes thro' the Center, is called the *Conjugate Axis*.

Defin. 5. The Distance betwixt the Vertex of an Ellipse,

Ellipse, and the Section of the Transverse, and any Ordinate, is called an *Abscissa* thereof.

Defin. 6. If a Cone be cut by two Planes, one any how thro' its Vertex and Center of the Base, making the Triangle ABD, (*Fig.* 24.) and another at right Angles to this, and parallel to one Side of the said Triangle, the Figure of the Section is a curved Line called a *Parabola*, the common Section of the parabolic and vertical Plane ABD, *viz.* VO, is the Axis thereof, and V the Section of a Side of the Triangle ABD and parabolic Plane, the *Vertex* of the Section.

Defin. 7. The Parts of Lines drawn in the Parabola at right Angles to the Axis, intercepted betwixt the same and Curve, are called *Ordinates* of the Axis.

Defin. 8. The Distance on the Axis, betwixt the Vertex and any Ordinate, is the *Abscissa* thereof.

Defin. 9. If opposite Cones be cut by two Planes, one any how thro' the Vertex and Center of the Base, making the Triangles ABD, A*bd*, (*Fig.* 25.) and the other at right Angles to this, so as to cut the opposite Cones (on both Sides the Vertex A) the Figure of the Sections made in the Conic Surfaces, are curved Lines called *Opposite Hyperbola's*; and each of them an *Hyperbola*. Let V, *v*, be the Sections of the Cutting-Plane with AB, AD the Sides of the Triangle ABD produced, then V *v*, is called the *Transverse Axis* of each Hyperbola, V, *v* its *Vertexes*; and VO the Section of the Plane ABD, and Hyperbola, is called its *Axis*.

Defin. 10. The Parts of Lines drawn within the Hyperbola, at right Angles to the Axis, intercepted by the same and Curve, are called *Ordinates* of the Axis.

Defin. 11. The Distance on the Axis betwixt the Vertex and Section of the Axis and any Ordinate,

dinate, is called the *Abscissa* thereof. Hence in the Ellipse and Hyperbola, every Ordinate has *two* Abscissa's, and in the Parabola but *one*.

Defin. 12. The *Ellipse*, *Parabola*, and *Hyperbola*, are called the *Conic Sections*. For tho' the Triangle and Circle may also be had by cutting a Cone, (as has been already observed) yet each of these having other Descriptions, and Names previously assigned them, the Ancients gave the Name of *Conic Section*, only to the three Figures mentioned in this Definition.

Defin. 13. In the Figures 23, 24, 25, let ABD be the Section of the Cone, by a Plane thro' the Vertex and Center of its Base, V*v* the Axis of the Sections, which (produced if needful) meets the common Section of the Base and vertical Plane ABD, (*viz.* BD) in O; and let a Line be drawn thro' V, parallel to BD, meet AD in G; then a fourth Proportional to the Lines VO, OB, and VG, is called the *Parameter*, or *Latus Rectum* of the Section, whether Ellipse, Parabola, or Hyperbola. Whence if p be put for the Parameter, then $VO : OB :: VG : p = \frac{VG \times OB}{VO}$.

Defin. 14. The Plane ABD, which in every Section is drawn thro' the Vertex and Center of the Cone's Base, and perpendicular to the Conic Section, we will for Brevity sake call the *Vertical Plane* of that Section.

Prop. I.

In every Ellipse it will be as the Transverse Axis is to the Parameter, so is the Rectangle of the Abscissa of any Ordinate to the Square of that Ordinate.

Let ABD (*Fig.* 23.) be the Section of the Cone thro' its Vertex, and the Center of the Base, and

V*mvn*

$Vmvn$ the Ellipse given, whose Axis is Vv; take any Point o in the Axis, thro' which let a Plane pass parallel to the Base BMDN, meeting the Ellipse in the right Line mon, and the Triangle ABD in EF; draw GV parallel to BD, and continue Vv till it meet BD (the Section of the vertical and elliptical Planes) in O.

Then because both $Vmvn$ and $EmFn$ are perpendicular to ABD, their common Section (mn) will also be perpendicular to ABD. (18 *Euclid.* 11.) Therefore mn is perpendicular both to EF and Vv, whence mo and no are Ordinates of the Abscissæ Vo, vo, (*Defin.* 4. *Euclid* 11.) but $EmFn$ is a Circle, (by *Cor.* 2. *Defin.* 2.) therefore from the (8. 6. *Euclid*) $om^2 = Eo \times oF$; but the Triangles VoF, VOB are similar by Construction; therefore $Vo : oF :: VO : OB$, (4 *Euclid* 6.) hence $oF = \frac{Vo \times OB}{VO}$ (16 *Euclid* 6.) Again, the Triangles voE, vVG, are similar, therefore $vo : oE :: vV : VG$, whence $oE = \frac{vo \times VG}{vV}$, therefore $oE \times oF = \frac{Vo \times vo \times OB \times VG}{VO \times vV}$ but $Eo \times oF = \overline{om}|^2$, consequently $\overline{om}|^2 = \frac{Vo \times vo \times OB \times VG}{VO \times vV}$ but $\frac{VG \times OB}{VO} = p$, (by *Defin.* 13.) therefore $\overline{om}|^2 = \overline{on}^2 = \frac{Vo \times vo \times p}{Vv}$, or $Vv : p :: Vo \times vo : \overline{mo}|^2 = \overline{on}|^2$. Q. E. O.

Cor. 1. Therefore if the transverse Axis of an Ellipse is called t, any Ordinate om, or on, y, and either of the Abscissæ thereof, *viz.* vo, or VO, x, then by this Proposition we have $t : p :: \overline{t-x} \times x : y^2 = \frac{p}{t} \overline{xtx - x^2}$, which may be called the Equation of the Ellipse.

Cor.

Cor. 2. In this by making $x = \frac{t}{2}$, and calling the conjugate Axis c, then y becomes equal to $\frac{c}{2}$; whence we have $\frac{c^2}{4} = \frac{p}{t} \times \frac{t^2}{2} - \frac{t^2}{4}$, or $p = \frac{c^2}{t}$, put this for p in the above expression, and we find $y^2 = \frac{c^2}{t^2} \times \overline{tx - x^2}$, which is a *second Equation of the Curve.*

Cor. 3. Hence the Parameter of the Axes of every Ellipse is a third Proportional to the transverse and conjugate Axes.

Cor. 4. If a Circle be described on the transverse Axis of any Ellipse, and an Ordinate be drawn to it, also cutting the Ellipse, then it will be as the transverse Axis is, to the Conjugate, so is the circular Ordinate to the elliptic Ordinate; that is (*Fig.* 33.) $Vv : MM :: NO : nO$, for from the Property of the Circle $VO \times Ov = \overline{ON}^2$; and by this *Proposition* $VO \times Ov : \overline{nO}^2 :: \overline{Vv}^2 : \overline{MM}^2$, hence $\overline{NO}^2 : \overline{nO}^2 :: VO \times vO \times \overline{Vv}^2 : VO \times vO \times \overline{MM}^2$, that is $NO : nO :: Vv : MM$.

PROP. II.

In every Conic Parabola it will be as the Parameter is to any Ordinate, so is that Ordinate to its Abscissa.

Let ABD (*Fig.* 24.) be the vertical Plane, or Section of the Cone, through its Vertex, and the Center of its Base, MVN the parabolic Plane whose Axis is VO, the Vertex being V; in the Axis, assume any Point o, through which let a Plane be drawn parallel to the Base, whose Section with the parabolic Plane is the right Line mon, with the Triangle ABC, the right Line EF, and with the Conic Surface, the Circle

EmFn,

$EmFn$, whose Diameter is EF, draw VG parallel to BD, then 'tis manifest from what was said in the last Proposition; mn is perpendicular both to EF and VO, therefore $EmFn$ being a Circle $Eo \times oF = \overline{om}|^2$, but the Triangles VoF, VOB are equiangular, whence $VO : OB :: Vo : oF$, therefore $oF = \frac{OB \times Vo}{VO}$, but $Eo = GV$, because they, as also EG and Vo are parallel, whence $oF \times oE = \frac{OB \times Vo \times GV}{VO}$, but $p = \frac{OB \times GV}{VO}$ (*per Defin.* 13.) whence $oF \times oE = p \times Vo$, but $oF \times oE = \overline{mo}|^2 = \overline{on}|^2$, therefore $\overline{om}|^2 = p \times VO$, that is $p : om :: om : Vo$. *Q. E. D.*

Cor. 1. Therefore $p \times VO = \overline{OM}|^2$, hence $VO : Vo :: \overline{MO}|^2 : \overline{mo}|^2$, that is in Parabola, *the Absciss's are as the Squares of their Ordinates.*

Cor. 2. Let Vo be called x, and om, y, then from this *Proposit.* $px = y^2$, which may be called the *Equation of the Parabola.*

Cor. 3. Hence, if thro' any Point of the parabolic Curve, a Line be drawn parallel to the Axis thereof, thereby cutting the Base unequally, *the Rectangle of the Parts of the bisected Base, will be equal to the Rectangle of the said parallel Line, and the Parameter of the Parabola.*

For (*Fig.*44.) draw the Ordinate vw, then by this *Proposition* $p \times VO = \overline{OM}|^2$, and $p \times Vw = \overline{vw}|^2$, hence $p \times \overline{VO - Vw} = \overline{MO}|^2 - \overline{vw}|^2$, that is $p \times vo = \overline{MO + vw} \times \overline{MO - vw} = NoxoM$.

PROP. III.

In every Conic Hyperbola it will be as the transverse Axis is to its Parameter, so is the Rectangle of the Abscissa's of any Ordinate to the Square thereof.

Let ABD and A*bd* (*Fig.* 25.) be the Section of the opposite Cones by a Plane thro' their common Vertex A, and the Axis of the Cone meeting the Bases thereof in the right Lines BD, *bd*, and let MVN denote the Section of the Cone, by a plain Perpendicular to ABD, which being produced, would also cut the Triangle A*bd* in *v*, then MVN *mvn* are Hyperbola's, whose Vertexes are V, *v*, and the transverse Axis V*v*: in VO the Hyperbolic Axis assume any Point *o*, thro' which let a Plane be drawn parallel to the Base of the Cone, meeting the hyperbolic Curve VMN in the right Line *mon*, the Triangle ABD in the right Line EF, and the Conic Surface in the Circle E*m*F*n*: also draw VG parallel to EF or BD: then again *om* or *on* will be equal and perpendicular both to the Axis VO, and the Diameter EF, whence *om* and *on* are each Ordinates of the Abscissæ VO, *vo*; (whence by the 8. 6. *Eucl.*) $Eo \times oF = \overline{mo}^2 = \overline{on}^2$; but the Triangles VOB, V*o*F, are similar; and so are *vo*E and *v*VG, therefore VO : OB :: V*o* : *o*F; therefore $oF = \frac{Vo \times OB}{VO}$; also *v*V : VG :: *vo* : *o*E, hence $oE = \frac{vo \times VG}{vV}$, consequently $oE \times oF = \overline{mo}^2 = \overline{on}^2 = \frac{Vo \times ov \times OB \times VG}{vV \times VO}$, but $\frac{OB \times VG}{VO} = p$, therefore $\overline{mo}^2 = \overline{on}^2 = \frac{Vo \times ov \times p}{Vv}$, that is $Vv : p :: Vo \times ov : \overline{mo}^2 = \overline{on}^2$. *Q. O. E.*

Cor.

Cor. 1. Therefore call the transverse Axis of the Hyperbola, *viz.* $Vv=t$, the Abscissa Vo, x, and the Ordinate $mo=on$ $(=)$ y, then $vo=t+x$, whence $y^2=\frac{p}{t}\times\overline{t+x}\times x$, which may be called the *Equation of the Hyperbola.*

From these Properties of the Conic Sections we might deduce a great Number of others, for they are the Foundation of all the rest; also by the help of a few Lemma's the Elliptic or Hyperbolic Spaces might be compared together, as is done by that great Promoter of Mathematicks in *France*, the late Marq. *de l'Hospital*, translated into *English* by my ingenious Friend Mr. *Edm. Stone*, known to the mathematical World by the Translations he has made of several curious Treatises from the *Latin* and *French*, to which I shall refer for any other Properties of the Conics I may want in the following Work, which the Limits I have prescribed my self, would not allow room for here. But before I finish this Chapter, it will not be amiss to give the Method of describing these Curves *in Plano*, since some may be desirous not only to measure them and the Solids thence form'd, but also to give a Description thereof.

Of describing the Conic Sections in Plano.

PROBLEM I.

The Parameter of any Parabola being given, to describe the same in Plano.

Draw two indefinite Lines AB, CD, (*Fig.* 26.) perpendicular to each other, in any Point V, assumed for the Vertex of the Section; then on the second (CD) from V, set off VP equal to the Parameter given, and from P assume any Number of

of Points c, c, c, c, c, on which with the Distances cP, cP, cP, *&c.* describe the Circles Pss, Pss, Pss, *&c.* cutting the Line AB in the Points s, s, s, s, s, s, *&c.* and the Line CD in the Points P, and o, o, o, o, *&c.* thro' which draw Lines parallel to AB; then if thro' each of the Points s, s, s, s, *&c.* Lines are drawn parallel to CD, they will cut those parallel to AB, in m, m, m, m, *&c.* Points of the Curve required; which connected with a regular Curve-Line, it will be the Parabola sought.

For 'tis manifest (8 *Eucl.* 6.) the several Lines Vs are mean Proportionals betwixt PV, and the corresponding Line Vo; therefore if the Parameter VP is called p, any of the Lines Vo, x, and its corresponding om, y, we have $px=y^2$, which is the Equation of the Curve described, whence mVm is a Parabola, *Cor.* 1. *Proposit.* 2. Q. E. F.

By this Construction a Parabola may be described with great facility by help of a parallel Ruler; and is also applicable, when the Parameter of any other Diameter as well as that of the Axis is given, provided the Angle made by that Diameter, and its Ordinates is also known.

Cor. 1. Hence if the vertex Axis and one Point of a Parabola be given, the same may be described: for from thence the Parameter may be found by joining the corresponding Points s, o, and bisecting the Line so perpendicularly; for that will meet the Axis produced in c, the Middle betwixt the Points P and o.

Problem II.

The Transverse and Conjugate Axis of an Ellipse being given, to describe the same in Plano.

Draw the two indefinite Lines AB, CD, (*Fig.* 27.) perpendicular to each other, in the latter of which

which assume any Point V for the Vertex of the Ellipse, and from V on CD set VP equal to the Semiconjugate Diameter, and on P with the Distance PV describe a Semicircle *sVs*; also through P draw any Number of Lines P*r*, P*r*, P*r*, &c. cutting the Circumference in as many Points *s*, *s*, *s*, *s*, &c. and the Line AB in the same Number of Points *r*, *r*, *r*, *r*; then from O, the Middle of the Transverse, or Center of the Section, draw Lines to the several Points *r*. Lastly, If through the several Points *s*, *s*, *s*, &c. Lines are drawn parallel to VO*v* the Axis, they'll cut the corresponding Lines O,*r*, O,*r*, O,*r*, &c. in the Points *m*, *m*, *m*, *m*, &c. which are the Points of the Ellipse sought: Or if MM be the conjugate Axis, and MR is drawn perpendicular to it, let MP be taken equal to the Semitransverse, and a Circle described thereon, with the Distance PM, then proceed as above to find the Points *m*, *m*, *m*, &c. And this is not only a very ready Method for denominating any Ellipse when its Axes are given, but extends to those Cases when any two conjugate Diameters are given, and the Angle they make with each other. The Reason of this Construction is easy, for thro' any of the Points *s*, and its corresponding *m*, draw Lines parallel to AB, let the first meet CD in *q*, and the second in *o*; call the transverse *t*, conjugate *c* and put Vo=x, also mo=sq=y, then Pq= $\frac{\sqrt{c^2-4y^2}}{2}$, and the similar Triangles Psq, PrV gives $\frac{\sqrt{c^2-4y^2}}{2} : y :: \frac{c}{2} : \frac{cy}{\sqrt{c^2-4y^2}}$ = Vr; also from the similar Triangles Qom, and QVr, we have $\frac{t}{2}-x$: $y :: \frac{t}{2} : \frac{ty}{t-2x}$=V$r$, hence $\frac{ty}{t-2x}=\frac{cy}{\sqrt{c^2-4y^2}}$ or

$\frac{t}{t-2x} = \frac{c}{\sqrt{c^2-4y^2}}$, or $t^2 \times \overline{c^2-4y^2} = \overline{t-2x}|^2 \times c^2$, that is $t^2c^2 - 4t^2y^2 = t^2c^2 - 4tc^2x + 4c^2x^2$, whence $y^2 = \frac{c^2}{t^2} \times \overline{tx-x^2}$, which is the Equation of the Ellipse, (*Cor.* 2. *Prop.* 1.) therefore the Curve described is an Ellipse whose transverse is t, and conjugate c. Q.E.F.

PROBLEM III.

The Parameter and Transverse Axis of an Hyperbola being given, to describe the same in Plano.

Draw the Lines AB, CD, (*Fig.* 28.) cutting each other perpendicularly in the Point V, taken for the Vertex of the Hyperbola; then on CD take VC equal to half the transverse Axis, and on the contrary Side AB, take VP equal to half the Parameter; on CP as a Diameter describe a Circle CRPR cutting AB in the Points R, R, through which, and C, draw two indefinite right Lines CR*r*, CR*r*, and through the Points V, let any Number of right Lines be drawn, cutting the Lines CR, CR, in the Points *r*, *r*, *r*, *r*, *&c.* and *s*, *s*, *s*, *s*, *&c.* Thence if from the several points *r* the corresponding Distances V*s*, V*s*, V*s*, V*s*, *&c.* be laid along the Lines *r*V, *r*V, *r*V, *r*V, *&c.* you will have the Points *m*, *m*, *m*, *m*, *&c.* of the Hyperbola sought.

For through any point *s* and its corresponding *r*, and *m*, draw the lines *sq*, *rw*, and *mo*, as also *mn* parallel V*w*; then because V*s*=*rm* by construction, and the Angle *s*V*q*=*r*V*w*, the Lines *rn* and *mn* will be respectively equal to *sq* and *q*V, for the Angles at *n* and *q* are Right. Now 'tis manifest the Triangles CVR and C*wr* are equiangular,

angular, whence if the transverse is called t, Parameter p the Line RR, c and $Vo=x$; also $mo=y$ and $qV=mn=e$, we have $t:c::\frac{t}{2}+x+e:rw$; therefore $rw=\frac{c}{2t}\times\overline{t+2e+2x}$; but the equiangular Triangles Cqs and CVR gives $t:c::\frac{t-2e}{2}:qs$, therefore $qs=\frac{c}{2t}\times\overline{t-2e}=rn$, whence $mo=y=\frac{c}{t}\times\overline{2e+x}$, but the Triangles Vsq and Vom are equiangular; thence $e:\frac{c}{2t}\times\overline{t-2e}::x:y$, therefore $ey=\frac{cx}{2t}\times\overline{t-2e}$; from this Equation we have $2e=\frac{ctx}{ty+cx}$; and from the former Equation $2e=\frac{ty-cx}{c}$, therefore $\frac{ty-cx}{c}=\frac{ctx}{ty+cx}$, or $c^2tx=t^2y^2-c^2x^2$; therefore $y^2=\frac{c^2}{t^2}\times\overline{tx+xx}$, but it is evident from the Construction $c^2=tp$, therefore $y^2=\frac{p}{t}\times\overline{tx+x^2}$, which is the Equation of the Hyperbola, (*per Cor. Prop.* 3.) therefore the Curve described is an Hyperbola, whose transverse Axis is t, and Parameter p. Q. E. F.

By this Method a Parabola might be described, when any Diameter and its Parameter are given, together with the Angle form'd by the said Diameter and its Ordinates. But what has been said is sufficient for our purpose, therefore now we shall proceed to shew how from these Propositions, and the following Lemma, those Spaces may be measured which are included by any of the Conic Sections.

CHAP. III.

Of the Method of measuring Curve-lined Spaces, with the Measures of the Circle and its Segments, as also the Conic Sections.

IN the first Chapter we have shewn the Method of measuring all kinds of right-lined Spaces, which originally depends on the *fourth Proposition* of the *first Book of* Euclid, where is inferred the Equality of two Triangles by their coinciding in all their Parts when mutually applied to each other.

But since all Figures, of what kind soever, are measured by the Ratio which they have to the measuring Unit, (See *Defin.* 23. *Ch.* 1.) a Figure contained wholly by right Lines, it follows from that Principle only, the Measures of Figures contained by Curves cannot be had; because no right-lined Figure, tho' equal in Area to a curve-lined one, can exactly correspond with it in all its Parts, and therefore some other Principle besides that given for right-lined Figures is necessary. An Expedient formerly brought for this purpose (by *Archimedes* under the Name of the Method of *Exhaustions*, since greatly improved by *Cavallerus* and Dr. *Wallis*) was to suppose the given Figure, whether Plane or Solid, divided into an infinite Number of Parts by Lines or Planes, either all parallel to one another, or all issuing from a given Point (as in measuring the several kinds of Spirals, *&c.*) these Parts may properly be called the *Elementa* of the Figures; then these Elementa being very small, were measurable from the Principles delivered already for right-lined Figures, because, tho' contain'd partly by right Lines, and partly by Curves, yet the curved Part being very small, it may be esteem'd a streight Line; and the Ele-

menta

menta it self equal to another similar to it, which is wholly contain'd by right Lines, or (if a Solid) plane Surfaces; whence after this, nothing further was wanted, but a Method of summing these Elementa, or their Measures. Which, however difficult it may appear at first Sight, *viz.* to sum a Series consisting of an infinite Number of Terms, yet in many Cases 'tis an easier Problem, than when their Number is finite.

The most Illustrious *Newton* consider'd this Matter after a different manner; for instead of the Elementa themselves, he substituted the Velocities by which they may be supposed generated. But this is a Consideration too far fetch'd for Beginners in these Speculations, I shall therefore confine my self to the former, and give the Principles and Uses thereof in as clear and general a Method, as the Nature of the Thing seems to admit of. In order to which, we premise this

LEMMA.

Let $\overline{0}|^{m}$, $\overline{a}|^{m}$, $\overline{2a}|^{m}$, $\overline{3a}|^{m}$, $\overline{4a}|^{m}$, *&c.* .... $\overline{na}|^{m}$ *be an infinite Series of the* m *Powers of Quantities in arithmetic Progression from* 0 *to* na *inclusive, where the Number of significant Terms is the infinite Number* n; *then shall* (s) *the Sum of them all be express'd by* $\frac{\overline{a}|^{m}}{m+1} \times \overline{n}|^{m+1} = \frac{n}{m+1} \times \overline{an}|^{m}$, *or putting the last Term* l, *viz.* $\overline{an}|^{m} = l$, then $s = \frac{nxl}{m+1}$.

DEMONSTRATION.

For if $s = \frac{a^m}{m+1} \times n^{m+1}$, when the Number of Terms is n, that when this is diminished by l, (the

 last

last Term) or the Number of Terms is $n-1$, shall be $s-l=\frac{am}{m+1}\times\overline{n-1}\rceil^{m+1}$, (*viz.*) by writing $s-l$ for s, and $n-1$ for n; but $\overline{n-1}\rceil^{m+1}=n^{m+1}-\frac{m+1}{1}n^{m}+\frac{m+1}{1}\cdot\frac{m+0}{2}n^{m-1}-\frac{m+1}{1}\cdot\frac{m+0}{2}\cdot\frac{m-1}{3}n^{m-2}$ &c. therefore $v-l=\frac{a^m}{m+1}\times\overline{n^{m+1}-\frac{m+1}{1}\cdot n^{m}+\frac{m+1}{1}\cdot\frac{m+0}{2}\cdot n^{m-1}}$ — &c. Let this be taken from $s=\frac{\overline{a}\rceil^m}{m+1}\times\overline{n}\rceil^{m+1}$, and there remains $l=\frac{a^m}{m+1}\times\overline{\frac{m+1}{1}\cdot n^{m}-\frac{m+1}{1}\cdot\frac{m+0}{2}\cdot n^{m-1}+\frac{m+1}{1}\cdot\frac{m+0}{2}\cdot\frac{m-1}{3}\cdot n^{m-2}}$, &c. But since by Supposition n is a Numb r infinitely great, therefore the first Term of the above Value of l, must be infinitely greater than any of the subsequent ones; hence all of them, but the first, may be neglected, and then we find $l=\frac{\overline{a}\rceil^m}{m+1}\times\frac{m+1}{1}n^m$, or $l=\overline{an}\rceil^m$, as it is by supposition, whence the Sum was rightly assigned. *Q. E. D.*

But since $l=\overline{an}\rceil^m$, let l be put for $\overline{an}\rceil^m$ in the above Value of s, and we have $s=\frac{n\times l}{m+1}$; *that is the Sum of an infinite Series of (the* m*) Powers of Quantities in Arithmetic Progression from* 0, *is equal to the Product of the last Term, by the Number of Terms, and this divided by the Index* (m) *plus Unity.*

Scholium. This Lemma contains the whole of Dr. *Wallis's Arithmetic of Infinites*, which he has established by an Inductionary Method; its Use in squaring of Curves, and cubing Solids, shall be explained in the following Propositions.

But

But we must observe from the same Principle by which we have given the Sum of a Series of Powers consisting of an infinite Number of Terms, we might deduce that when they are finite, and then the former will only be a particular Care of the latter.

For let $p^m + \overline{p+r}|^m + \overline{p+2r}|^m + \overline{p+2r}|^m \ldots \overline{p+nr}|^m$ be a Series of Powers, whose Roots are in arithmetic Progression, and put

$$A = \frac{r^m}{m+1}$$

$$B = r^{m-1} p + \frac{m+1}{2} A$$

$$C = \overline{r^{m-2} p^2 - \frac{m+1}{3} A + B} \times \frac{m}{2}$$

$$D = \overline{\frac{m}{3} \cdot r^{m-3} p^3 + \frac{m+1}{3} \cdot \frac{m}{4} A - \frac{m}{3} B + C} \times \frac{m-1}{2}$$

$$E = \overline{\frac{m}{3} \cdot \frac{m-1}{4} r^{m-4} p^4 - \frac{m+1}{3} \cdot \frac{m}{4} \cdot \frac{m-1}{5} \cdot A +}$$
$$\overline{\frac{m}{3} \cdot \frac{m-1}{4} \cdot B - \frac{m-1}{3} C + D} \times \frac{m-2}{2}$$

$$F = \overline{\frac{m}{3} \cdot \frac{m-1}{4} \cdot \frac{m-2}{5} r^{m-2} p^5 + \frac{m+1}{3} \cdot \frac{m}{4} \cdot \frac{m-1}{5} \cdot \frac{m-2}{6}}$$
$$\overline{A - \frac{m}{3} \cdot \frac{m-1}{4} \cdot \frac{m-2}{5} \cdot B + \frac{m-1}{3} \cdot \frac{m-2}{4} C - \frac{m-2}{3}}$$
$$\overline{D + E} \times \frac{m-3}{2}$$

&c.

Then S, the Sum of all the Terms shall be express'd by this Theorem, *viz.*

$$S = An^{m+1} + Bn^m + Cn^{m-1} + Dn^{m-2} \ldots + p^m$$

Where it is to be observed, the Terms A, B, C, *&c.* must be found till you have got that belonging

 ing

ing to the Term where the Index of r is 1; and when that is a Fraction, the Series is infinite; But if $p=o$, then these Terms are express'd thus, *viz.*

$$A = \frac{r^m}{m+1}$$

$$B = r^m \times \tfrac{1}{2}$$

$$C = r^m \times \frac{m}{3 \cdot 4}$$

$$D = o$$

$$E = -r^m \times \frac{m \cdot \overline{m-1} \cdot \overline{m-2}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$

$$F = o$$

$$G = r^m \times \frac{m \cdot \overline{m-1} \cdot \overline{m-2} \cdot \overline{m-3} \cdot \overline{m-4}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}$$

&c. *&c.*

As found by my ingenious Friend Mr. *Tho. Sympson*, in a late Treatise on the Laws of *Chance*. But the above Theorem for summing of Series of Powers, has lain by me some Years, nor should I ever have made it publick, had not this Occasion offered, because the differential Method of Sir *Isaac Newton* is better fitted for summing Series's, and which has been largely insisted on by Mr. *James Stirling*, in a whole Treatise on that Subject.

PROP. I.

The Measure of every Sector of a Circle is equal to half the Product of the Measure of a Radius thereof, into the Arch which is the Base of the Sector.

For let CAE (*Fig.* 29.) be the Sector whose Center is C, and Base the Arch AE, which suppose divided into an infinite Number of Parts in the Points *a*, *b*, *c*, *d*, *&c.* and draw the Chord A*a*, on

on which let fall the Perpendicular CP, which will be infinitely near equal to CA, the Radius of the Sector: put r for the Measure of CA, and q the Measure of ($Aa=ab=ac=ad$, &c.) one of the Parts into which the Base of the whole Sector is divided, and l the Measure of the Base of the Sector, *viz.* $l=AE$, then $\frac{l}{q}$ is the Number of the Elementa CAa, Cab, Cbc, Ccd, &c. (for $q : 1 :: l : \frac{l}{q}$) but 'tis evident they are all equal, hence $CAa \times \frac{l}{q}$ is the Measure of the whole Sector, but the Measure of CAa is $\frac{r \times q}{2}$ (*Cor.* 2. *Prop.* 1. *Chap.* 1.) hence the Measure of the whole Sector is $\frac{r \times q}{2} \times \frac{l}{q} = \frac{r \times l}{2}$. *Q. E. O.*

Corol. 1. If on C with any other Radius as R an Arch of a Circle be described, cutting CA, CE in α, ϵ, then the Area of the latter Sector shall be to that of the former, as R^2 : is to r^2; for $\frac{\alpha\epsilon \times R}{2}$ is the Area of the Sector $C\alpha\epsilon$, and $r : l :: R : \frac{lR}{r} = \alpha\epsilon$, per Figure; therefore $\frac{l \times R^2}{2R}$ is the Area of the latter, and $\frac{r \times l}{2}$ is the Area of the former: but $\frac{l \times R^2}{2r} : \frac{l \times r}{2} :: R^2 : r^2$. *Q. E. D.*

Cor. 2. Whence the Measure of every Circle is equal to the Product of a Radius thereof into half the Circumference. Therefore before the Measure of a Circle can be had, we must shew how to find its Circumference from the Diameter being given.

PROP.

Prop. II.

Let t *be the Tangent of* BE (Fig. 30.) *any Arch of a Circle* (*not more than* 45 *Degrees*) *whose Center is* C, r *the Radius thereof; then shall the Arch* BE *be expressed by the following Series*, $t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}}$, &c. *infinitely.*

For let the Tangent BT be conceived divided into an infinite Number of equal Parts in the Points a, a', a'', a''', &c. to which from the Center let right Lines be drawn and produced, cutting the Arch in the Points e, e', e'', e''', e'''', &c. also on the Center C with the several Distances Ca, Ca', Ca'', Ca''', &c. describe the Arches Tv, vv', vv'', &c. cutting the produced Lines in as many Points v, v', v'', v''', &c. now 'tis evident the Arch Tv is infinitely small, and so may be taken for a right Line; also the Triangles TBC and Tva may be taken for Equiangular; therefore put q to denote the Measure of any are of the equal Parts Ta, aa', aa'', aa''', &c, then $\frac{t}{q}$ is their Number. Now from the 4th *Euclid* 6. we have CT : CB : : Ta : Tv, that is $\sqrt{r^2+t^2} : r : : q : \frac{rq}{\sqrt{r^2+t^2}} =$ Tv, but the Triangles CvT, and CEe are equiangular; Therefore CT : Tv : : CE : Ee, that is $\sqrt{r^2+t^2} : \frac{rq}{\sqrt{r^2+t^2}} : : r : \frac{r^2q}{r^2+t^2} =$ Ee, therefore by Division E$e = q \times \overline{t - \frac{t^2}{r^2} + \frac{t^4}{r^4} - \frac{t^6}{r^6} + \frac{t^8}{r^8}}$, &c. See the Process

$r^2+t^2)$

$$r^2+t^2)\, r^2\, (1-\frac{t^2}{r^2}+\frac{t^4}{r^4}-\frac{t^6}{r^6},\ \&c.$$

$$\frac{r^2+t^2}{-t^2}$$

$$-t^2-\frac{t^4}{r^2}$$

$$+\frac{t^4}{r^2}$$

$$\frac{t^4}{r^2}+\frac{t^6}{r^4}$$

$$-\frac{t^6}{r^4}$$

$$-\frac{t^6}{r^4}-\frac{t^8}{r^6}$$

$$+\frac{t^8}{r^6}$$

After the same manner calling Ba, Ba', Ba'', Ba''', &c. t, t', t'', t''', &c. successively, you will find

$$e\,e'=\frac{qr^2}{r^2+{}'t^2}=q\times\overline{1-\frac{{}'t^2}{r^2}+\frac{{}'t^4}{r^4}-\frac{{}'t^6}{r^6}+\frac{{}'t^8}{r^8}}\ \&c.$$

$$e'e''=\frac{qr^2}{r^2+{}''t^2}=q\times\overline{1-\frac{{}''t^2}{r^2}+\frac{{}''t^4}{r^4}-\frac{{}''t^6}{r^6}+\frac{{}''t^8}{r^8}}\ \&c.$$

$$e''e'''=\frac{qr^2}{r^2+t'''^2}=q\times\overline{1-\frac{{}'''t^4}{r^2}+\frac{{}'''t^4}{r^4}-\frac{{}'''t^6}{r^6}+\frac{{}'''t^8}{r^8}}\ \&c.$$

From hence 'tis manifest the Length of the Arch BE $=$ Ee $+$ $e\,e'$ $+$ $e'e''$ $+$ $e''e'''$, &c. is equal to the Sum of the following Series, contained both ways to Infinity.

viz.

viz. BE=

$$\left.\begin{array}{l}
q\times\overline{1-\frac{t^2}{r^2}+\frac{t^4}{r^4}-\frac{t^6}{r^6}+\frac{t^8}{r^8}-\frac{t^{10}}{r^{10}}}\ \&c.\\
q\times\overline{1-\frac{t'^2}{r^2}+\frac{t'^4}{r^4}-\frac{t'^6}{r^6}+\frac{t'^8}{r^8}-\frac{t'^{10}}{r^{10}}}\ \&c.\\
q\times\overline{1-\frac{t''^2}{r^2}+\frac{t''^4}{r^4}-\frac{t''^6}{r^6}+\frac{t''^8}{r^8}-\frac{t''^{10}}{r^{10}}}\ \&c.\\
q\times\overline{1-\frac{t'''^2}{r^2}+\frac{t'''^4}{r^4}-\frac{t'''^6}{r^6}+\frac{t'''^8}{r^8}-\frac{t'''^{10}}{r^6}}\ \&c.\\
\text{Infinitely.}
\end{array}\right\}\ \text{Infinitely.}$$

Now since the Number of Terms in each vertical Row is $\frac{t}{q}$ by what was said above, therefore the Sum of $\overline{1\times1\times1\times1\times1}$, &c. $\times q = \frac{t}{q}\times1\times q = t$; and the second is a Series of Squares, whose Roots t, t', t'', t''', &c. are in arithmetic Progression, the greatest being $\frac{t^2}{r^2}$, hence in the preceding Lemma for l,

n,

n, and m, writing $\frac{t^2}{r^2}$, $\frac{t}{q}$ and 2 respectively, we have the Sum of $q \times \overline{\frac{t^2}{r^2} + \frac{{}^{\prime}t^4}{r^4} + \frac{{}^{\prime\prime}t^6}{r^6}}$, &c. $= q \times \frac{t^2}{r^2} \times \frac{t}{q} \times \frac{1}{2+1} = \frac{t^3}{3r^2}$: After the same manner the Sum of the third Row will be $\overline{\frac{t^4}{r^4} + \frac{{}^{\prime}t^6}{r^6} + \frac{{}^{\prime\prime}t^8}{r^8}}$ &c. $\times q = q \times \frac{t^4}{r^4} \times \frac{t}{q} \times \frac{t}{4+1} = \frac{t^5}{5r^4}$, the Sum of the fourth Row will be $q \times \frac{t^6}{r^6} \times \frac{t}{q} \times \frac{1}{5+1} = \frac{t^7}{7r^7}$, and so on; whence connecting these Sums under their proper Signs for the Length of the Arch BE, we have $t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}}$, &c. *Q. E. I.*

It may be observed it was not necessary in this Process, to find more than the first horizontal Row of these Series, because each of that being the last or greatest Term in the several Powers, and the Form of all the rest the same; the whole might have been summed from the Lemma without further trouble; but we set the Values of the several Arches Ee, $e\,e'$, $e'e''$, $e''e'''$, &c. down for the sake of Beginners, because it must be more satisfactory to them, since having exhibited the Values of the several Elementa, their Sum must manifestly be equal to the whole Arch.

Cor. 1. Now let CB$=1$, and the Arch BE be 30 *Degr.* then its Tangent is manifestly equal to $\sqrt{\frac{1}{3}}$, for the Sine of 30 *Degr.* is $\frac{1}{2}$, *viz.* half the Chord of 60 *Degr.* which is equal to the Radius,

therefore the Cosine is equal to $\sqrt{1-\frac{1}{4}}=\frac{\sqrt{3}}{2}$; but as the Cosine of any Arch is to its Sign, so is the Radius of the Tangent thereof, that is $\frac{\sqrt{3}}{2} : \frac{1}{2} :: 1$ $\sqrt{\frac{1}{3}}$, hence BE $=\frac{1}{\sqrt{3}}-\frac{1}{3.\times3\sqrt{3}}+\frac{1}{5.9\sqrt{3}}-\frac{1}{7.27\sqrt{3}}+\frac{1}{9.81\sqrt{3}}-\frac{1}{11.243\sqrt{3}}$, &c. and six times this must be the Length of the Semicircumference; therefore put $6\sqrt{\frac{1}{3}}=\sqrt{12}=A$, $\frac{A}{3.3}=B$, $\frac{3B}{5.3}=C$, $\frac{5C}{7.3}=D$, &c. Then half the Circumference is equal to A,—B+C—D+E—F, &c. See the Process,

$\sqrt{12}$ = A=3.4641.0161.5
$\frac{B}{5}$ =C= . . 769.8003.6
&c. E= . . . 47.5185.4
G= . . . 3.6552.7
I= 3105.8
L= 279.4
N= 26.1
P= 2.5
R= 2
&c.

A+C+E+G+L = 3.5462.3317.2
4046.4051.7

3.1415.9265.5

$\frac{A}{9} = B =, .3849.0017.9$

$\frac{5C}{21} = D =, ..183.2858.0$

&c. $F = ...12.9596.0$

$H = ...1.0559.7$

$K = ...926.3$

$M = ...85.0$

$O = ...8.0$

$Q = ...8$

&c.

$B + D + F + H = 4046.4051.7$

The latter being taken from the former, leaves 3.14.15.9265.5 for the Length of half the Circumference of a Circle whose Radius is Unity: *Therefore the Diameter of any Circle is to its Circumference as* 1 *is to* 3.1415.9265.5 *nearly.*

There are various other Ways of finding the Circumference of a Circle, particularly one given by Mr. *Machin*, and published in the *Synopsis Palmariorum Mathefeos*, of that incomparable Mathematician Mr. *Jones*, where is shewn that the Diameter of a Circle being 1, the Circumference will be $\frac{16}{5} - \frac{4}{239} - \frac{1}{3} \times \frac{16}{5^3} - \frac{4}{239^3} + \frac{1}{5} \times \frac{16}{5^5} - \frac{4}{239^5}$ *&c.* a few Terms of which will exhibit the same Number we found above. But my Friend, to whose Assistance I am so much indebted in this Performance, has favoured me with a Method by which the Solution above may be much abbreviated: For after any Number of Terms of the above Series is found, if you put n, the Index of t in the last of them, which we will call T, then the Sum of the remaining Terms shall be $+ \frac{n \times Tt^2}{\overline{n+2} \times r^2 + t^2}$

+

$+\frac{2T't^2}{\overline{n+4}.\overline{r^2+t^2}}+\frac{2.2T''t^2}{\overline{n+6}.\overline{r^2+t^2}}+\frac{2.3T'''t^2}{\overline{n+8}.\overline{r^2+t^2}}$
$+\frac{2.4T''''t^2}{\overline{n+10}.\overline{r^2+t^2}}+$ &c. Where we must observe the Sign of T' is ever contrary to that of T. the Terms T, T', T'', T''', &c. succeeding in Order; thus in the preceding Example, the Sum of the first six Terms, *viz.* $\overline{t-\frac{t^3}{3r^2}+\frac{t^5}{5r^4}-\frac{t^7}{7r^6}+\frac{t^9}{9r^8}}$
$\overline{-\frac{t^{11}}{11.r^{10}}}\times 6$, is equal to 3.1413.0878.6; and here $n=11$, and the sixth Term is $\frac{t^{11}}{11r^{10}}=$,0012.95960=T; hence youl'll find $\frac{11\times T\times t^2}{13\times\overline{r^2+t^2}}$

$$=\frac{11T}{52}=T'=\ ,0002.74145$$

$$\frac{1.2T'}{60}=T''=\ldots\ 9138$$

$$\frac{2.2T''}{68}=T'''=\ldots\ 537$$

$$\frac{3.2T'''}{76}=T''''=\ldots\ 43$$

$$\frac{4.2T''''}{84}=T'''''=\ldots\ 4$$

Their Sum + ,0002.83867

To which the first six Terms being added, *viz.* 3.1413.08786, we have for the Semi-circumference 3.1415.92653 the same as was found by 17 Terms of the original Series; whereas by this Method we had but 11, and if more initial Terms had been col-

collected, we had still obtained the Sum of the remaining ones by a less Number of these in Proportion.

Cor. 2. Hence if D be the Diameter of any Circle, and a=,7854, then since 1 : 3,1416 (=$4a$) : : D : $4a$D the Circumference of the Circle whose Diameter is D, and therefore (by *Cor.* to the last *Proposit.*) $\frac{D}{2} \times \frac{4aD}{2} = aD^2$ is the Area thereof.

Whence we have the following *Rule to measure any Circle whose Diameter is given.* Multiply the Square of the Diameter by ,7854, and the Product is nearly the Area thereof.

Cor. 3. If C be the Circumference of any Circle, then since 3.1416=$4a$: 1 : : C : $\frac{C}{4a}$ the Diameter thereof; therefore by the same *Corol.* $\frac{C}{2} \times \frac{C}{8a} = \frac{C^2}{16a}$ is the Area thereof, which gives this *Rule for Measuring any Circle whose Circumference is given.*

Divide the Square of the Circumference by 12.5664, and the Quotient is (nearly) the Area thereof. And from hence we have the two following *Rules to measure any Circle by the Sliding-Rule.*

Rule 1. Set 1 on D to ,7854 on C, then opposite the Diameter on D is the Area on C.

Rule 2. Set 1 on D to ,079578 on C, then opposite the Circumference on D is the Area on C.

Or set 3.5449 on D to 1 on C, then opposite the Circumference on D is the Area on C.

Examp. Let the Diameter of any Circle be 20, and its Area required, the Square of the Diameter

is - - - -	20x20=400	
Which multiply'd - by	0,7854	
And the Product is -	314,1600	the Area thereof.

By the Sliding-Rule, set 1 on D to ,7854 on C, then opposite 2 on D (for 20 is off the Rule) is 3.1416 on C; and because 20 was divided by 10, I multiply 3,1416 by 100, the Square of 10, and find 314.16 for the Area as before. (See *Prop.* 5. *Sect.* 2. *Part* I.)

But if 20 be multiplied by 3.1416, we have 62.832 for the Circumference; hence by the second Rule above, I set 1 on D to ,079 on C, then opposite 3.14 (*viz.* the $\frac{1}{20}$ of 62,832) on D, is ,7854 on C; which I multiply by 400, the Square of what the Circumference was divided by, and find 314,16 for the Area as before. Or by the second Method 3,5449 on D set to one on C, then against 6.2832 on D is 3.1416 on C, which multiply'd by 100 (for a manifest Reason) gives 314,16 as before.

Corol. 4. Hence if 180 Degrees is equal in Length to 3.1415926 *&c.* so is one Degree to $\frac{3,1415926}{180}$ = ,01745329, the Length of one Degree in such Parts as the Radius is 1; therefore ,01745329 × *r* is the Length of one Degree when the Radius is *r*; consequently the Length of *d Degrees* is ,01745329 × *rd*, and this multiplied by $\frac{r}{2}$ gives ,008726647 × r^2 × *d* for the Area of a Sector whose Radius is *r*, and the Base consists of *d Degrees*; which may be expressed thus,

The Number of Degrees in any Sector multiplied by ,008726647, *and that Product by the Square of its Radius, so shall this last Product be the Area of the Sector.*

Or, if Unity be divided by 008726647, and the Square Root of the Quotient be extracted, it will be found 10,7048; therefore if 10,7048 on D is set to the Degrees in the Sector's Base on C, then

then oppoſite the Radius (of the Sector) on D, is the Content on C.

Examp. Let the Radius of the Sector be 62.5,and Baſe 80 *Degr.* then $62.5 \times 62.5 \times 80 \times 008726647 =$ 2727,08 nearly the Area. Or 10.7048 on D ſet to $\frac{80}{4}$ on C, then oppoſite 62.5 on D is 681.77 on C; and this multiplied by 4, becauſe 80 was divided thereby, gives 2727.08, the ſame as before. (See *Prop.* 5. *Sect.* 2. *Part.* 1.)

Cor. 5. From the laſt *Corol.* is deduced the Rule for meaſuring the Segment of a Circle; for if from the Area of the Sector, be taken the Area of a Triangle, two of whoſe Sides are its Radii, and the third the Chord of its Baſe, there will remain the Area of the Segment; now put $q=,0174532$, and s the Sine of d *Degrees* in the Table of *Natural Sines*, then $r \times s$ is its Sine when the Radius is r, hence $rs \times \frac{r}{2} = \frac{s \times r^2}{2}$ is the Area of the ſaid Triangle; therefore $\frac{qdr^2 - r^2 s}{2}$ is the Area of the Segment, which gives the following Rule to meaſure any Segment of a Circle.

Let the Number of Degrees in the Segment be multiplied by ,0174533 and the natural Sine of thoſe Degrees (from the Tables) taken from the Product; then half the Remainder multiplied by the Square of the Radius of the Segment is the Area thereof.

Examp. Let us reaſſume the laſt Example, where the Arch was 80 Degrees: its Sine in the Tables is 9848077; and 80 multiplied by 0174533 is 1,3962560, from which ,9848077 being taken, there remains ,411483; half this multiplied by $62.5 \times 62.5 = 3906,25$ gives 803,61, nearly the Area of the Segment.

This may be done at two Operations by the Sliding-Rule; for let the Area of the Sector be

found by the laſt *Corol.* then ſet 1 on D to half the Sine of the Segment's Arch from the Tables on C, and oppoſite the Radius on D is the Area of the Triangle on C, ſo in this Example 1 on D ſet to ,4924038 againſt $\frac{62.5}{20}=3.125$ on D (for 62.5 is off the Rule) is 4,8086, which multiplied by 400 the Square of the Diviſor of 62.5 gives 1923,44; this taken from 27.27, the Sector's Area, leaves 803,64 for the Area of the Segment, nearly the ſame as before. But there ſeems ſtill wanting a Method of computing the Area of any Segment of a Circle without the Arch, that being ſeldom a Part of the Data in Practice, therefore we ſhall now lay down a *New Method* for that Purpoſe; for the Reader by purſuing the ſame Steps with thoſe for obtaining the Arch from its Tangent, might get the common Series for the Segment; but in place of that Series, which is to be met with in moſt Authors that have wrote on the Quadratures of Curves, I ſhall preſent the Reader with a new one, both converging *faſter* and more *ſimple*.

PROP. III.

Let C (*Fig.* 31.) be the Center of the Segment EBDE, whoſe verſed Sine is BG $= v$, the Radius CB $= \frac{d}{2}$, and $V = d - v$, then the Area of the Segment EBDE ſhall be expreſſed by this Series,

$$4\sqrt{vV}\times\frac{v}{1.3}+\frac{v^2}{1.3.5V}-\frac{v^3}{3.5.7V^2}+\frac{v^4}{5.7.9V^3}-\frac{v^5}{7.9.11V^4}$$

&c. or put A for the firſt Term, *viz.* $A=\frac{4v}{3}\sqrt{vV}$, B the ſecond, C the third, then the Area of the Segment is $A+\frac{Av}{5V}-\frac{1Bv}{7V}+\frac{3Cv}{9V}-\frac{5Dv}{11V}+\frac{7Ev}{13V}-$&c.

The

The Demonstration of this Series depends on Principles not yet publickly known, so the Reader must take it on trust till my Friend shall think proper to publish not only that, but several other useful Improvements he has made in this Business of Quadratures, as well as several things in the Method of Infinite Series, and the other recluse Parts of Mathematicks.

But we must not pass over that useful Approximation given for the Area of the Segment of a Circle by the Great *Newton*, and publish'd by Dr. *Wallis*, and the most accurate Mr. *Jones*; and also by Mr. *Colson*, in his late Version of *Newton's Fluxions*, in the last of which is its Demonstration; the Theorem is this: If the Diameter of any Circle be d, the versed Sine of any Arch v, then the Area of a Segment contain'd by twice the said Arch, and twice its Sine, will be $\overline{\sqrt{dv} + 4\sqrt{dv - \frac{3}{4}v^2}} \times \frac{4v}{15}$, which may be expressed in Words thus; Let the Diameter be multiplied by the versed Sine of any Segment, and the Product called the first Number, from which substract three fourths of the Square of the said versed Sine, and call the Remainder the second Number, then the Square Root of the first Number added to four times the Square Root of the second, and that Sum multiply'd by four fifteenths of the versed Sine, will be nearly the Area of the Segment sought. Or we may give this Approximation for the Area of the Segment of a Circle which in most Cases is nearer than that above, *viz.* $\frac{80d - 39v}{16d - 3v}\sqrt{dv} \times \frac{4v}{15}$ is the Measure of the Segment of the Circle, whose Diameter is d, and versed Sine v.

We will illustrate both these Methods by an Example: Let the Diameter of any Circle be 50

Inches, and the versed Sine of any Arch thereof 10 Inches, then $\frac{v}{V}=\frac{10}{40}=\frac{1}{4}$, also $\frac{4v\sqrt{vV}}{3}=$ $\frac{4\times10\sqrt{400}}{3}=\frac{800}{3}=266,6666$ &c. $=A$ also $B=\frac{A}{4\cdot5}$, $C=\frac{B}{4\cdot7}$, $D=\frac{3C}{4\cdot9}$, $E=\frac{5D}{4\cdot11}$, $F=\frac{7E}{4\cdot13}$, &c. See the Operation.

A =	266,66667	C =	47619
B =	13,33333	E =	451
D =	3968	G =	9
F =	61		
		—	.48079
+	280.04029		
—	48079		
	279.55950 the Area of the Segment.		

Now if we make use of *Newton*'s Rule, $\sqrt{dv}=\sqrt{500}=22.36069$, and $4\sqrt{dv-\frac{3}{4}v^2}=4\sqrt{425}=82,46212$, the Sum of these is 104.82281, and this multiplied by $\frac{4v}{15}=\frac{8}{3}$ gives 279,52749; nearly the same as before.

Or by the second Approximation $\frac{4000-300}{800-30}\sqrt{500}\times\frac{40}{15}=\frac{361}{77}\sqrt{500}\times\frac{8}{3}=279,557$; differing from the truth only ,002. which is less than the $\frac{1}{139500}$th of the whole Segment.——And this Method is also easier in practice.

By any of these Methods the Area of a circular Segment may be found, but in Practice a Table of the Areas of the Segments of a Circle, whose Diameter is divided into 1000 equal Parts is preferable; therefore since such a Table is so very useful (as will appear farther on) in finding the Quan-

Quantity of Liquor drawn out, or remaining in a Cask not full, we shall proceed to shew the manner of computing it by this Proposition.

Let the Diameter of a Circle be 1, which suppose divided into 1000 equal Parts, thro' every one of which imagine Perpendiculars drawn and continued both ways to the Circumference.

Then since the first versed Sine is ,001, we have $v=,001$ and $\frac{v}{V}=\frac{,001}{999}=\frac{1}{999}$; therefore $A=\frac{4v}{3}\sqrt{vV}=\frac{4}{3000}\sqrt{,000999}=00004214$, and $B=\frac{Av}{5V}=\frac{00004215}{999\times5}=00000001$, whence the first Number of the Table is ,00004215.

The second versed Sine is $,002=v$, therefore $V=,998$ and $\frac{v}{V}=\frac{2}{998}=\frac{1}{199}$, whence $A=\frac{8}{3000}\sqrt{,001996}=,00011914$, and $B=\frac{Av}{5V}=\frac{00011914}{499\times5},00000005$; therefore the second Number is ,00011919.

The third versed Sine is $,003=v$, therefore $V=1-,003=,997$, whence $\frac{v}{V}=\frac{3}{997}$, and $A=\frac{12}{3000}\sqrt{002991}=,00021876$: also $B=\frac{Av}{5V}=\frac{,00021876\times3}{5\times997}=00000013$, thence the third Number of the Table is ,00021889.

And by this Method the whole might be computed, but after a sufficient Number of Terms are found at the beginning of the Table, the rest may be had (to 7 or 8 Places of Decimals) by this

 Rule:

Rule: Let $\alpha, \beta, \gamma, \delta$, denote any four Terms succeeding in the Order of the Letters, then $\delta = \alpha + 3 \times \overline{\gamma - \beta}$.

And if any Term of this Table be divided by 78539816, or multiplied by its Reciprocal, you'll have the common Table of Segments when the Area is Unity; but the first Form is preferable, as will appear further on.

But to shew the Use thereof, we must lay down this

THEOREM.

If two Circles have the same Center, and from thence be drawn two Lines to any two Points in the outer Circumference, thereby cutting the inner one in the same Manner, and the Chords of those Arches be drawn, forming a Segment in each; then shall the versed Sines of these Segments be to one another, as their respective Radii; and the Segments as the Squares thereof.

Let C (*Fig.* 31.) be the Center of the outermost Circle, and E, D, the two Points taken in its Circumference, which are join'd by the Lines CE, CD; also let *e b d* be part of another Circle described on the same Center, which cuts the Lines CE, CD in the Points *e*, *d*; draw the Lines ED, *e d*, and the Line CB to B. a Point in the middle betwixt the Points E, D; let it cut the Arch *e d* in *b*, and the Lines ED, *e d* in G, *g* respectively; then I say GB : *g b*, so is CD to *c d*; also EBDE : *e b d e* : : $\overline{CD}^2 : \overline{Cd}^2$. For in the first place CD : CG : : C*d* : C*g*, that is CD : CD—GB : : C*d* : C*d* —*bg*, therefore (by 17 *Eucl.* 5.) CD : GB : : *cd* : *gb*. Again, let us put S for the Sector CDE, *s* the Sector C*de*, also T the Segment DBED, and *t* the Segment *d b e d*, then (by the 19 *Euclid* 6.) $\overline{CD}^2$: $\overline{Cd}^2$: : S—T : *s*—*t*, but (*Cor.* 1. *Prop.* 1.) S : *s* : : $\overline{CD}^2$:

$\overline{CD}|^2 : \overline{Cd}|^2$, therefore $S : s :: S-T : s-t$, or (as above) $S : T :: s : t$, whence $\overline{CD}|^2 : \overline{Cd}|^2 :: T : t$. *Q. E. D.*

Corol. From this Theorem and the foregoing Table, we may easily compute the Area of any Segment from its versed Sine being given.

For if v be put for any versed Sine of a Circle whose Diameter is d, then by the first Part of this Theorem $d : v :: 1 : \frac{v}{d}$ and $1 : 1000 :: \frac{v}{d} : \frac{1000v}{d}$. Let this be look'd for in the Table of Segments under VS, whose Diameter is 1, and call the Number against it n, then by the second Part of the Theorem $\overline{1}|^2 : n :: d^2 : d^2 \times n$, which is the Area of a similar Segment, whose versed Sine is v, and Diameter d.

We may express this Method in Words thus: Let three Cyphers be annexed to the given versed Sine, and then divided by the Diameter of its Circle, seek the Quotient under VS, and take out the Number against it under the Letters *Seg.* Multiply this Number by the Square of the Diameter, and the Product is the Area of the Segment sought. I am afraid the Reader will think I have taken too much Pains about the Circle, its Sectors and Segments; but they are of such particular Use in Gauging, and ullaging of Casks, that I was willing to set this Affair in as clear a Light as possible, therefore I hope that will excuse my seeming Prolixity.

PROP. IV.

The Area of every Conic Parabola, is equal to two thirds of the Product of the Measures of its Abscissa and double Ordinate.

For let MVN (*Fig.* 32.) be a Conic Parabola, whose Vertex is V, Axis VO, and MON the double

double Ordinate of the Abſciſſa VO, conceive the Abſciſſa VO divided into an infinite Number of Parts in the Points o, o', o'', o''', *&c.* and put the Meaſure of one of them q, the Meaſure of Vo, x; and MN, y; alſo let the ſucceeding Abſciſſas Vo, Vo', Vo'', Vo''', *&c.* be x', x'', x''', x'''', *&c.* their correſponding Ordinates y', y'', y''', y'''', *&c.* and the Parameter of the Parabola p: Now ſince V$o = x$ is divided into an infinite Number of Parts, each of which is expreſſed by q, therefore $\frac{x}{q}$ is their Number: from *Prop.* 2. *Chap.* II. $4px = y^2$, therefore $y = 2\sqrt{px}$ and $q \times y = 2q \times \sqrt{px}$ the Meaſure of the firſt and greateſt Ellementa MmnN, after the ſame manner the ſecond will be $2q \times \sqrt{px'}$, the third $2q \times \sqrt{px''}$, *&c.* as here under, where for Brevity of Expreſſion we denote the ſeveral Elementa in order by e, e', e'', e''', *&c.*

$$\text{Hence}\left\{\begin{array}{l} e = 2q\sqrt{px} = 2q\sqrt{p} \times \sqrt{x} \\ e' = 2q\sqrt{px'} = 2q\sqrt{p} \times \sqrt{x'} \\ e'' = 2q\sqrt{px''} = 2q\sqrt{p} \times \sqrt{x''} \\ e''' = 2q\sqrt{px'''} = 2q\sqrt{p} \times \sqrt{x'''} \\ \&c. \qquad\qquad\qquad \&c. \end{array}\right.$$

Therefore the Area of the Parabola is equal to a Series of the $\frac{1}{2}$ Powers of Quantities (x, x', x'', x''', *&c.*) in Arithmetic Progreſſion from o, multiplied by $2q\sqrt{p}$, hence by the Lemma $2q\sqrt{p} \times x^{\frac{1}{2}} \times \frac{2}{3} \times \frac{x}{q} = \frac{4\sqrt{p} \times x^{\frac{3}{2}}}{3}$ is the Area; for the laſt or greateſt Term is $2q\sqrt{x}$, the Number of Terms (n) is $\frac{x}{q}$, and the Index (m) is $\frac{1}{2}$, hence $\frac{nl}{m+1}$ in the Lemma gives $\frac{x}{q} \times 2q\sqrt{px} \times \frac{2}{3} = \frac{4x\sqrt{px}}{3}$ but $\sqrt{px} = \frac{y}{2}$, there-

therefore $\frac{2x \times y}{3}$ is the Area of the Parabola. *Q. E. D.*

And from hence by the Sliding Rule we have this Solution:

Set $1\frac{1}{2}$ on A to the Abſciſſa on B, then oppoſite the double Ordinate on A is the Area on B.

Examp. Let the Abſciſſa VO be 124 Inches, the double Ordinate MN=135 Inches, then $\frac{2 \times 135 \times 124}{3}$ =11160 Square Inches the Area.

Or, ſet $1\frac{1}{2}$ on A to 12,4 (the $\frac{1}{10}$ Part of 124) on B, then oppoſite 13.5 (the 10th Part of 135) on A, is 111.6 on B: And becauſe the Terms 135, 124 were each divided by 10, multiply 111,6 by 100 (the Product of their Diviſors) and find 11160 for the Area as before.

Prop. V.

If on the Tranſverſe of an Ellipſe a Circle be deſcribed, and any Point taken in the Axis, and a Perpendicular drawn thro' the ſame, cutting of a circular and elliptic Segment; then ſhall the Area of the former Segment be to the latter, as the tranſverſe Axis is to the conjugate.

For (*Fig.* 33.) let VGvG be the Ellipſe, whoſe tranſverſe is Vv, and conjugate GG, cutting each other in C, the Center of the Ellipſe, and let VMvM be a Circle deſcribed on C, with the Diſtance Cv=CV; alſo take O any Point in Vv, through which let a Line be drawn perpendicular to the Axe, meeting the circumſcribing Circle in N, N, and the Ellipſe in *n*, *n*, then will the circular Segment N v N be to the elliptic Segment *n* v *n* as Vv is to GG. For conceive Ov to be divided into an infinite Number of equal Parts in the Points *o*, *o*, *o*, *o*, &c. thro' which draw Lines parallel

rallel to NN, cutting the circumscribed Circle in N′, N″, N‴, &c. and the elliptic Curve in n', n'', n''', &c. also put q to denote the Measure of one of the Parts $Oo = oo = oo$, now by the Property of the Ellipse (*Corol.* 4. *Prop.* 1. *Chap.* 2.) NN : nn : : Vv : GG ; therefore NN $\times q$: $nn \times q$: : Vv : GG ; that is the circular Elementa N N N′ N′, is to the elliptic Elementa $n\ n\ n'\ n'$: as the transverse Axis is to the conjugate ; and the same may be shewn (after this manner) of all the other Elementa ; wherefore (by 12 *Euclid* 5.) Vv : GG : : NvN : $n\,v\,n$. *Q. E. D.*

Cor. 1. Hence *the Area of every Ellipse is to the Area of its circumscribed Circle, as the conjugate Axis is to the transverse:* Therefore put a to denote the Area of the Circle, whose Diameter is Unity, then $\overline{Vv}|^2 \times a$, is the Area of the circumscribed Circle, (*Cor.* 2. *Prop.* 2.) therefore by this Proposition Vv : GG : : $\overline{Vv}|^2 \times a$: GG $\times v$V $\times a$ the Area of the Ellipse, which gives this Rule: *The Area of every Ellipse is equal to the Product of the Transverse by the conjugate Axis, and that Product by* .7854. *But for the Sliding-Rule* ; Let us find the Value of $\frac{1}{,7854}$, which is 1.27324, then set 1,27324 on B to the transverse on A, and opposite the Conjugate on B, is the Area of the Ellipse on A.

Examp. Let it be required to find the Area of an Ellipse whose transverse Axis is 61,6, and conjugate 44,4, then $44.4 \times 61,6 \times .7854 = 2148.1$ is the Area sought.

Or, *by the Sliding-Rule*, set 1,273 on B, to 61.6 on A, then opposite 4,44 on B (for 44.4 falls beyond A) is 214.81 on A, and this multiplied by 10, (for Reasons explained *Part* 1. *Sect.* 4. *Prop.* 3.) gives 2148.1 the same as before.

And hence from what was said about the Segment

ment of a Circle, 'twill be eafy (by *Prop.* 3. or the Table of *Seg.*) to find the Area of any Segment of an Ellipfe, whofe Abfciffa (or verfed Sine) is given.

PROP. VI.

Let $t = 2CV$ the tranfverfe Axis of the Hyperbola (*Fig.* 28.) and put x to denote Vo, any Abfciffa thereof whofe Ordinate om is y, then fhall the Area of mv in that Part whofe Abfciffa is x, be expreffed by the following Series where $V = t + x$,

viz. $\overline{4yx \times \frac{x}{1.3} - \frac{x^2}{1.3.5.V} - \frac{x^3}{3.5.7.V^2} - \frac{x^4}{5.7.9.V^3} - \frac{x^5}{7.9.11.V^4}}$ &c.

The Demonftration of this Series depends on the fame Principles with that for the Circular Segment *Pag.* 100. and therefore for the fame Reafon the Reader muft take the Inventor's Word for its Truth, till he fhall think fit to publifh its Demonftration.

But we muft not omit that ufeful Solution given for this Purpofe by Mr. *Coats*, *Prop.* IV. of his *Harmonia Menfurarum*, and fince publifhed by moft of the Authors that have wrote on Fluxions, particularly my ingenious Friend Mr. *Tho. Simpfon*; tho' I muft obferve there is a Prefs-Error in his Solution, as will appear by comparing it with the following one: but if I miftake not, we were firft obliged with a Geometrical Demonftration of the Property of the Hyperbolic Spaces (on which this depends) by that elaborate Geometer *Gregory St. Vincent.* The Theorem is this; If t be the tranfverfe Axis, and c the conjugate of any Hyperbola (*viz.* $c = \sqrt{tp}$) and x any Abfciffa whofe Ordinate is y, then the Area of the Hyperbola, whofe Abfciffa is x, will be expreffed by

$\frac{2 + 2x}{2}$

$\frac{t+2x}{2} \times y - t \times c \times ,575646 \times L. : \frac{t+2x+\frac{2ty}{c}}{t}$ For an Example to both these Methods, let $t=30$, $x=10$, $y=12$, then $c=\frac{ty}{\sqrt{tx+xx}}=18$; hence by the first Theorem we have this Process,

$$\frac{4xy}{3}=A=160.0000$$

$$\frac{Ax}{5V}=B=\ .8.0000$$

$$\frac{Bx}{7V}=C=\ldots 2857$$

$$\frac{3Cx}{9V}=D=\ldots. 238$$

$$\frac{5Dx}{11V}=E=\ldots\ldots 27$$

$$\frac{7Ex}{13V}=F=\ldots. 4$$

Sum —— 8.3126

The Area 151,6874

And by the second Method, $\frac{t+2x}{2} \times y=300$, also $\frac{t+2x+\frac{cty}{c}}{t}=3$, and its Logarithm is ,4771212, therefore $t \times c \times ,575646 \times L : 3 = 540 \times ,575646 \times ,4771212 = 148,3126$; this taken from 300, leaves 151,6874, the same as was found by the following Series, so that the one confirms the other.

Besides these two Methods for measuring the Hyperbola, we may add some Rules analogous to those for measuring the circular Segment: for using the same Letters as in the Proposition, the Measure of the hyperbolic Segment is nearly express'd by any of these three THEOREMS.

I.

$$\left.\begin{array}{ll}\text{I.} & \overline{\sqrt{tx}+4\sqrt{tx+\tfrac{3}{4}x^2}}\times\frac{4x}{15}\times\frac{c}{t}\\ \text{II.} & \overline{4\sqrt{tx}+21\sqrt{tx+\tfrac{5x^2}{7}}}\times\frac{4x}{75}\times\frac{c}{t}\\ \text{III.} & \ldots\ldots\ \frac{80t+39x}{16t+3x}\sqrt{tx}\times\frac{4x}{15}\times\frac{c}{t}\end{array}\right\}\begin{array}{l}=151.648\\=151.681\\=151.602\end{array}$$

The two firſt are taken from the *Analyſis per Quantitatem*, &c. of Sir *Iſaac Newton*, (and publiſh'd by Mr. *Jones*) but the ſeçond is neareſt the Truth.

And now we have done with meaſuring Planes, whether contained by Right Lines, or any of the Conic Sections; in the next place we proceed to Solids.

CHAP. IV.

Of Meaſuring Solids.

PROP. I.

THE Meaſure of every Solid, generated by a Line moving betwixt two parallel equal and ſimilar Planes ſimilarly ſituated (called the Ends thereof) is equal to the Meaſure of one of the Ends, multipled by the Meaſure of a Perpendicular, let fall thence on the other End.

For let b expreſs the Meaſure of that Perpendicular, and e the Meaſure of one End or Baſe; then if at the Diſtance of the linear Unit, Planes be drawn thro' the ſolid parallel to its Baſe, it is manifeſt that the ſeveral Elementa contained by every two of theſe contiguous Planes, will be equal to one another; therefore as often as the Baſe contains the *ſuperficial Unit*, ſo often will each Elementa contain

contain the *solid Unit*, but e expresses that Number. Again, as often as the Perpendicular contains the *linear Unit*, so often will the whole Solid contain e solid Units; but b is that Number, therefore $b \times e$ is the Number of solid Units, or Measure of the generated Solid by *Defin.* 23. *Chap.* 1. *Part.* 2. *Q. E. D.*

Definit. 1. If the Ends or Planes about which the revolving Line turns be two Squares, each of whose Sides is equal to the Distance of the Ends, and also the generating Line is perpendicular to both of them, the generated Solid is called a *Cube.*

2. If the Ends are Parallelograms, the generated Solid is called a *Parallelopipedon.*

3. If the Ends are Triangles or Polygons of any Number of Sides, the generated Solid is in general called a *Prism.*

4. If the Ends are Circles, the generated Solid is called a *Cylinder.*

5. If the Ends are any other Curve-Lined Spaces, we may call them *Cylindroids*, prefixing the Name of the Base. Thus let them be two Similar, and similarly posited Ellipses, then the Solid is an *Elliptical Cylindroid*; if Parabola's, a *Parabolic Cylindroid*; when Hyperbola's, an *Hyperbolic Cylindroid*; all which are measured by help of the above Proposition only, and the assistance of the Propositions foregoing.

Cor. 1. For if the generated Solid be a Cube whose Side AB (*Fig.* 34.) is d, then the Area of the Base, or End, is d^2, (*Prop.* 1. *Ch.* 1. *Part* 1.) therefore for e, b, in the above Proposition, write d^2 and d, and $e \times b$ becomes $d^3 = d \times d \times d$, whence this Rule for *Measuring the Cube.*

Multiply the Measure of the Side of the Cube by it self, and that Product again by the Measure

of the Side, so shall this last Product be the Measure of the Cube sought.

Or, *by the Lines* D, E *on the Sliding-Rule*, set 1 on D to 1 on E, then opposite the Side on D is its Measure on E.

Example. Let the Side of a Cube be 6 Inches,

$$\begin{array}{l} 6 = d \\ 6 \\ \hline 36 = d^2 \\ 6 \\ \hline 216 = d^3 \text{ the Measure sought.} \end{array}$$

Or, 1 on D set to 1 on E, then against 6 on D, is 216 on E, the Content as before; this might be done by the Lines C, D, for one on D set to 6 on C, against 6 on D, is 216, the Content on C.

Cor. 2. Hence if the generated Solid be a Parallelopipedon, whose Ends are right-angled Parallelograms, the Measures of whose Sides are l, b, and the Measure of a Perpendicular let fall from one End to the other (continued if needful) h, then $l \times b$, (by *Prop.* 1. *Chap.* I.) is the Measure of the Base; therefore for e in the Proposition, write $l \times b$, and we have $l \times b \times h$ for the Measure of the Parallelopipedon proposed.

Whence this Rule for measuring all Parallelopipedons, having rectangular Bases. Multiply the Product of the Width and Breadth of the Base by the Height of the Solid, and that Product will be its Measure.

Or this may be done by the Lines A, B, on the Sliding-Rule at two Operations. Thus 1 on B set to the Width on A, against the Breadth on B, is the Area or Measure of the Base on A; then 1 on B set to the Area of the Base (found above)

on A, againſt the Height of the Solid on B, is its Meaſure on A.

Examp. Let there be a Parallelopipedon ABCD, (*Fig.* 35.) whoſe Breath is AB=8 Inches, Width AE=6 Inches, and from ſome Point H in one End, let a Perpendicular be let fall on the other, let it meet the ſame in P, and ſuppoſe HP=12 Inches; then by the above Rule $8 \times 6 \times 12 = 576$ Inches is the Meaſure of the Solid. Or, 1 on B ſet to 6 on A, againſt 8 on B, is 48 on A; againſt which bring 1 on B, then againſt 12 on B is 576 on A, the Meaſure ſought.

Cor. Hence the Meaſure of every triangular Priſm, is equal to the Meaſure of a Perpendicular let fall from any Angle of its Baſe on its oppoſite Side, into half that Side, and this Product multiplied by the Height of the Priſm. Thus,

(*Fig.* 36.) Let ABD be a Priſm whoſe Baſe ABC is a Triangle, from an Angle of which as C, let fall a Perpendicular on AB, meeting the ſame in P, alſo let a Plane be drawn thro' any Line *k k* of the End D, perpendicular to the Baſe, and let it meet the ſame in KK, and from any Point H of *kk*, a Line being let fall perpendicular on KK, meeting the ſame in Q, then HQ is the Height of the Priſm: Whence $CP \times \frac{AB}{2} \times HQ$ is the Meaſure thereof, which is evident from this Propoſition, and *Cor.* 2. *Prop.* 1. *Chap.* 1.

Cor. 4. If the Baſe was any regular Polygon, let the Area thereof be found by *Corol.* 4. *Prop.* 1. *Chap.* I. then this multiplied by the Height of the Priſm, will be its Content.

Examp. Let the Baſe be a Pentagon whoſe Side is 50, the ſame with that *Page* 68. and let the Meaſure of a Line drawn from a Point in the End to the Baſe (*Fig.* 37.) be 12 Inches, then the Area

of

of the Base is 4301,175, and that multiplied by 12, gives 51614,1 square Inches for the Measure of the pentagonal Prism ABEF. Also if the Area of the Base was any Conic Section, its Area may be found by one of the three last Propositions; then this multiplied by the Height, will be the Content of that Prism.

Cor. 5. Hence also the Measure of every Cylinder is equal to the Square of its Diameter multiplied by ,7854, and that Product by the Height of the Cylinder: But to measure this by the Sliding-Rule, another Consideration must be brought; for if the Diameter of the Cylinder's Base is d, and its Height h, then $d^2 \times h, \times ,7854$ is its Content: Call that C, then $d^2 \times h \times ,7854 = C$, or

$$d^2 \times h = \frac{C}{.7854} = C \times \overline{1.128378}\,|^2, \text{ for } \sqrt{\frac{1}{,7854}} =$$

1.128378, hence $\overline{1.128378}\,|^2 : h :: d^2 : C$. This Proportion gives this following *Rule for measuring a Cylinder by the Sliding-Rule.*

Set 1.128378 on D, to the Height of the Cylinder on C, then opposite the Diameter of its Base on D, is the Content on C. See *Prop.* 5. *Sect.* 4. *Chap.* 3. *Part* 1.

Examp. Let the Diameter AB (*Fig.* 38.) of the Cylinder's Base be 12 Inches, and its Height 16, to find the Content.

First, By the above Solution we have the following Operation.

$d = \text{-----------} \; 12$

$\quad\quad\quad\quad\quad\quad\quad\; 12$

$d^2 = \text{---------} \; 144$

$\quad\quad\quad\quad\quad\quad\quad\; 0{,}7854$

$$\begin{array}{r} 31416 \\ 31416 \\ 7854 \\ \hline \end{array}$$

$d^2 \times ,7854 = \text{- -} \; 113{,}0976$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 16$

$d^2 \times b \times ,7854 = C = 1809{,}5616$

Or by the Sliding-Rule, I set 1.128 on D, to 16 on C, then opposite 1.2 on D is 18,096 on C: and because 12 was divided by 10, I multiply 18,096 by 100, and have 1809.6 for the Measure of the Cylinder.

PROP. II.

Let VMN (Fig. 39.) *be a Curve of such a kind, that the Proportion of every Part* VO=x, *and the Perpendicular* MN *(passing thro'* O, *and intercepted by the Curve) being equal to* y, *be expressed by the Equation* $y^2 = Ax + Bx^2$ *the Letters* A, B, *denoting given Quantities; then if this Curve revolve round* VO *as an Axis (which from the Equation bisects the Lines* MN, *mn*, *m'n'*) *the Measure of the Solid generated by this Motion will be express'd by this* THEOREM, $\overline{\frac{Ax^2}{2} + \frac{Bx^3}{3}} \times a$,

(where a=78539816, *&c.)*

For let the Abscissa VO be conceived divided into an infinite Number of equal Parts in the Points *o*, *o'*, *o''*, *o'''*, *o''''*, *&c.* and thro' the same imagine Planes, to pass at right Angles to VO, the

the Axis of Revolution: Then 'tis manifest the several solid Elementa *Mn*, *m'n*, *m'n''*, &c. may be taken for so many Cylinders having equal Altitudes, the Diameters of their Bases the Lines MN, *mn*, *m'n'*. *m''n''*, &c. put q to denote the Measure of one of the infinitely small Parts O*o*, *oo'*, *o.o''*, &c, and let us denote the Measures of the successive elementary Cylinders M*n*, *mn'*, *m'n''*, &c. by e, e', e'', e''', &c. respectively, and x, x', x'', x''', &c. for the Distance of each them from V, *viz.* x, x', x'', x''', &c. $=$ V*o*, V*o'*, V*o''*, V*o'''*, &c. hence $\frac{x}{q}$ is the Number of elementary Cylinders, or Abscissas x, x', x'', x'''.

Now since $y^2 = Ax + Bx^2$, therefore $a \times y^2 = a \times \overline{Ax + Bx^2}$ is the Area of the Base of the elementary Cylinder M$n = e$; therefore by *Cor.* 5. of the last Proposition $q \times a x y^2 = q \times a \times \overline{Ax + Bx^2} = e$, in which for x write successively x', x'', x''', &c. and for e write e', e'', e''', &c. then we shall have the following Table of Elementary Cylinders, and their Values.

$$viz. \begin{cases} e = q \times a \times \overline{Ax + Bx^2} \\ e' = q \times a \times \overline{Ax' + Bx'^2} \\ e'' = q \times a \times \overline{Ax'' + Bx''^2} \\ e''' = q \times a \times \overline{Ax''' + Bx'''^2} \end{cases}$$
&c. &c.

Therefore the Measure of the generated Solid is equal to the Sum of a Series of the (1) *one* Powers of Quantit. in Arithm. Progression from o to Ax inclusive, more by the Sum of a Series of the (2) or *second* Powers of Quantit. in Arithm. Progression from o to Bx^2, and this Sum multiplied by $q \times a$. Now the Sum of the first, *viz.* $Ax + Ax' + Ax'' + Ax'''$, &c. to o, is $\frac{Ax}{2} \times \frac{x}{q}$, for by the Lemma *Pag.* 85. the last Term is Ax, the Number of Terms is $\frac{x}{q}$, and

the Index 1, whence for n, l, and m, write $\frac{x}{q}$, Ax and 2, and we have $\frac{Ax^2}{2q}$ for the Sum required. Also the Sum of $Bx^2 + Bx'^2 + Bx''^2 + Bx'''^2$, &c. to o, is $\frac{Bx^3}{3q}$ by writing for n, l, m, in the Lemma $\frac{x}{q}$, Bx^2, 2; hence then the Measure of the whole Solid, which we will call S, is $\overline{\frac{Ax^2}{2q} + \frac{Bx^3}{3q}} \times a \times q$, therefore $\overline{\frac{Ax^2}{2} + \frac{Bx^3}{3}} \times a = S$. *Q. E. O.*

Cor. 1. If in this Expression for Bx^2 be put its Value from the Equation of the Curve ($y^2 = Ax + Bx^2$) we shall find $S = \overline{Ax + 2y^2} \times \frac{ax}{6}$, the Measure of the Solid generated by the Motion of the Curve VMN about VO.

Cor. 2. If any other Abscissa as VC be called b, and its corresponding Ordinate PQ, b, then $\overline{\frac{Ab^2}{2} + \frac{Bb^3}{3}} \times a$ is the Solidity of the Conoid VPQ, whence putting F for the Measure of the Frustum MNQP, we have $F = 3A \times \overline{b^2 - x^2} + 2B \times \overline{b^3 - x^3} \times \frac{a}{6}$; but $b^3 - x^3 = \overline{b - x} \times \overline{b^2 + bx + x^2}$, and $b^2 + x^2 = \frac{b^2 + v^2 - A \times \overline{b + x}}{B}$ from the Equation of the Curve, whence

$$F = \overline{A \times \overline{b + x} + 2Bbx + 2b^2 + 2y^2} \times \frac{a}{6} \times \overline{b - x},$$ which is better accommodated to measuring the Frustums of Conoids, and consequently better fitted to Gauging.

From this Proposition, and the 2 *Cor.* above, we may easily deduce the Measure of the Cone, Sphere,

Sphere, Spheroids of both kinds, and the two Conoids and their Frustums: But first we must shew what is meant by these Names.

Defin. 1. If the generating Curve VMN (*Fig.* 43.) be a Circle, the generated Solid will be a Sphere, any part of which V*mn* (cut off by a Plane *mn*) from the Vertex, is called the *Segment of a Sphere*; and the difference of this from half of the Sphere, *viz.* MN*mn* is called a *Frustum of the Sphere*.

2. If the generating Curve VM*v*N (*Fig.* 42.) be an Ellipse and the Axe of Revolution the transverse Axis thereof, the generated Solid is called a *Spheroid*; but when the Axe of Revolution is the conjugate Axis, an *oblate Spheroid*, any Part of which V*mn* cut off by a Plane perpendicular to the Axis of Revolution, is called a *Segment* of the Spheroid; and what this wants of half the Spheroid, is called its *Frustum*; if the Spheroid is cut by a Plane parallel to the Axis of Revolution, we may call the Piece cut off a *Parallel Segment*; and any Part of the Segment of a Spheroid, cut off by a Plane perpendicular to its Base, is called a *second Segment* thereof.

3. If the generating Curve VMN (*Fig.* 41.) be a Parabola, the generated Solid is called a *Parabolic Conoid*; any Part from the Vertex V, (*viz.* V*mn*) cut off by a Plane perpendicular to VO the Axe of Revolution, is called its Segment; and what this wants of the whole Conoid, (*viz.* *mn*MN) is called its *Frustum*.

4. If the generating Curve VMN (*Fig.* 39.) be an Hyperbola, the generated Solid is called an *Hyperbolick Conoid*; and any Part of this from the Vertex, cut off by a Plane perpendicular to the revolving Axe, is called its *Segment*, and what this wants of the whole Conoid, is called an *Hyperbolic Frustum*.

5. But if the Circle or any of the Conic Sec-

tions revolve round an Ordinate to the Axis of the Circle, or Conic Section, the Solid in general is called a *Spindle*, with the Name of the generating Curve prefix'd. Thus if the generating Curve be a

Circle,	The generated Solid is a	*Circular*	Spindle.
Ellipse,		*Elliptic*	
Parabola,		*Parabolic*	
Hyperbola,		*Hyperbolic*	

Cor. 5. From the preceeding Propositions it follows, if the Height of any Cone (*viz.* VO *Fig.* 40.) be h, the Diameter of its Base (*viz.* MN) $= b$, then the Measure thereof shall be $ab^2 \times \frac{h}{3}$. And if this Cone be cut thro' o, any Point in the Axis by a Plane parallel to its Base, so that the Measure of Vo be x, and that of the Line mn, y, then the Measure of the Frustum mn M N is $\overline{b^2+by+y^2} \times a \times \frac{h-x}{3} = \frac{b^3-y^3}{b-y} \times \frac{a \times \overline{h-x}}{3}$.

For let the Cone be generated by the right-angled Triangle VON, round VO as an Axis, then *per* similar Triangles $x : y :: h : b$, therefore $hy = bx$, or $y^2 = \frac{b^2x^2}{h^2}$, this being compared with the Equation in the Proposition (*viz.* $y^2 = Ax + Bx^2$) gives $A = o$, $B = \frac{b^2}{h^2}$; therefore $S = \overline{Ax+2y^2} \times \frac{ax}{6}$ (*Corollary* 1.) is $S = \frac{y^2 \times ax}{3}$; this when x becomes h, and $y = b$, is $S = \frac{b^2 \times a \times h}{3}$; the Measure of the Cone AMN. Also $F = \overline{A \times \overline{h+x} + 2Bhx +} \overline{2b^2+2y^2} \times \overline{h-x} \times \frac{a}{6}$ becomes $F = \overline{\frac{2b^2}{h^2} \times hx + 2b^2} \overline{+2y^2} \times \frac{\overline{h-x} \times a}{6}$, or $F = \overline{b^2+by+y^2} \times \frac{\overline{h-x} \times a}{3}$ (for $\frac{2b^2x}{h}$

$\frac{2b^2x}{b} = 2by$, from the Equation of the Cone) and from hence the following Rules for measuring any Cone, and its Frustum universally having a circular Base, both by the Pen and Sliding-Rule.

Rule 1. *To measure any Cone, having a circular Base.*

The Square of the Cone's Base being multiply'd by its Height, and that Product by ,26179939, or in Practice by 2618, the Result will be the Measure sought.

Rule 2. *To measure any Cone by the Sliding-Rule.*

Set 1.9544 on D to the Cone's Height on C, then opposite the Diameter of the Base on D is the Content of the Cone on C.

For an Example of these two Rules: Let the Hieght of a Cone be 50 Inches, and the Diameter of its Base 30.

The Operation.

$b =$	30	,2618
	30	$45000 = b^2 \times b$
$b^2 =$	900	13090000
$b =$	50	10472
$b^2 \times b =$	45000	$11781{,}0000 = b^2 \times b \times {,}2618$

Or, by the Sliding-Rule 1.9544 on D being set to 50 on C, then against 3 on D (for 30 is of the Line) is 117.81; which multiplied by 100, the Square of the Divisor of 30 (see *Prop.* 5. *Sect.* 2. *Chap.* I.) gives 11781, for the Measure of the Cone as before.

Rule 3. *To measure the Frustum of a Cone by the Pen.*

To the greater Diameter of the Frustum and the Lesser; and from the Square of this Sum, take the Product of the two Diameters; multiply the Re-

Remainder by the Height of the Frustum, and that Product by ,2618 ; so shall this last Product give the Measure of the Frustum sought.

Rule 4. *To measure the Frustum of a Cone by the Sliding-Rule.*

Set 2,764 on D to the Height of the Frustum on C, then find both the greater and lesser Diameters, and also their Difference on D, and note the three Numbers opposite them on C: then from triple the Sum of the first and second, take the third, and the Remainder is the Measure sought.——The Reason of this Rule is from hence, that $b^2+by+y^2=\frac{3b^2+3y^2-\overline{b-y}|^2}{2}$.

Example of a Frustum of the above Cone. Let the Height of the Frustum be 20 Inches, then the Diameter is 18 Inches.

The Operation.

30	$=b$
18	$=y$
48	$=b+y$
48	
384	
192	
2304	$=\overline{b+y}\vert^2$
540	$=b\times y$
1764	$=\overline{b+y}\vert^2-b\times y$
20	$=b$
35280,0	$=\overline{\overline{b+y}\vert^2-b\times y}\times b$
8162	$=,2618$ inverted
70560	
21168	
353	
282	
Meas. of Conic Fru. 9236,3	$=\overline{\overline{b+y}\vert^2-bxy}\times b\times,2618$

Or

Or, by the Sliding-Rule 2,764 on D being set to 20 upon C, against 30,18 and 12 upon D is 2356.2; 848,23 and 377 respectively: the Sum of the first and second is 3204,43, three times which is 9613,3; from which the third being taken, leaves 9236.3 as before.

Cor. 4. Also from this Proposition it follows, if the Height of any parabolic Conoid be VO$=h$, (*Fig.* 41.) the Diameter of its Base b, and in the Axis any Point o be taken, distant from V by x, and a Plane be drawn thro' the same, whose Section with the Conoid is the Circle mn, the Diameter of which is $mn=y$, then the Measure of the Conoid VMN will be $\frac{ab^2\times h}{2}$; and the Measure of the Frustum mnMN$=\overline{b^2+y^2}\times a\times\frac{h-x}{2}$. For (*per Cor.* 1. *Prop.* 2. *Chap.* 2.) $h:b^2::x:y^2$, therefore $y^2=\frac{b^2}{h}\times x$, this compared with the Equation $(y^2=Ax+Bx^2)$ in the Proposition gives $A=\frac{b^2}{h}$, $B=o$, whence $S=\overline{Ax+2y^2}\times\frac{ax}{6}=\overline{\frac{b^2}{h}\times x+2y^2}\times\frac{ax}{6}$, but $\frac{b^2}{h}=y^2$, hence $S=3y^2\times\frac{ax}{6}=\frac{y^2\times ax}{2}$, and this when x becomes h is $\frac{b^2\times a\times h}{2}=$VMN; also $F=\overline{A\times\overline{h+x}+2Bhx+2b^2+2y^2}\times a\times\frac{h-x}{6}$ becomes $\overline{\frac{b^2}{h}\times\overline{h+x}+2b^2+2y^2}\times a\times\frac{h-x}{6}$, or $F=\frac{\overline{b^2+b^2x+2b^2+2y^2}}{h}=\frac{\overline{3b^2+3y^2}\times a\times\overline{h-x}}{6}=\overline{b^2+y^2}\times a\times\frac{h-x}{2}=mn$MN.

And from hence we have the following Rules for measuring the parabolic Conoid and its Frustum.

Rule

Rule 1. *To measure a parabolic Conoid by the Pen.*

Let the Square of the Conoid's Base be multiplied by its Height, and this Product by ,3927, the last Product is the Measure of the Conoid proposed.

Rule 2. *To measure a parabolic Conoid by the Sliding-Rule.*

Set 1.59577 on D, to the Conoid's height on C, then against the Diameter of its Base on D, will be the Measure sought on C.

Rule 3. *To measure the Frustum of a parabolic Conoid by the Pen.*

To the Square of the greater Diameter add the Square of the Lesser, and multiply their Sum by the Height; this Product multiply'd by .3927, is the Measure sought.

Rule 4. *To measure the Frustum of a parabolic Conoid by the Sliding-Rule.*

Set 1.59577 on D to the Conoid's Height on C, and find the greater and lesser Diameters on D, the Sum of the Numbers opposite them on C, is the Measure of the Frustum required.

For an Example to these four Rules; there is a parabolic Conoid, whose Height is 60 Inches, the Diameter of the Base 40 Inches, and which is cut at the Distance of 21.6 Inches from its Vertex by a plane Parallel to its Base, to find the Content both of the Conoid VMN, and its Frustum *mn*MN*mn* : it will be found that $mn=y=24$ for $60 : 40 \times 40 :: 21.6 : 24 \times 24 = \overline{mn}^2$, thence $mn = 24$, and $oO = 38.4$ the Height of the Frustum.

By the *first Rule* $s = \frac{a \times b^2 \times b}{2}$.

And by the *third Rule* $F = \overline{b^2 + y^2} \times a \times \frac{b-x}{2}$.

The

The OPERATION.

For the whole Conoid.

40	$=b$
40	
1600	$=b^2$
60	$=b$
96000,0	$=b^2 \times b$
7293	$=3927$ Inverted.
288000	
86400	
1920	
672	
37699,2	$=b^2 b \times ,3927$

For the Frustum.

$y=24$	
24	
96	
48	
$y^2=576$	

40	$=b$
40	
1600	$=b^2$
576	$=y^2$
2176	$=b^2+y^2$
38,4	$=b$
8704	
17408	
6528	
83558,4	$=\overline{b^2+b^2 \times b}$
7293	$=,3927$ Inverted.
250675	
75203	
1670	
585	
22813,3	$=\overline{b^2+b^2} \times b \times ,3927$

For the whole Conoid by the Sliding-Rule thus, by *Rule* 2. I set 1.59577 on D, to 40 on C, then against 6 on D (for 60 is off the Rule) is 376.99 on C; this multiply'd by 100, the Square of the Divisor of 60, gives 37699, the Measure of the Frustum as before.

To measure the Frustum of the Conoid by the Sliding-Rule, thus by *Rule* 4. I set 1.59577 on D to 38.4

38.4 on C; then against 2.4 and 4 on D, is 86.86 $=\alpha$ and 241.27 $=\beta$, the Sum of which is $\alpha+\beta=328.13$, which I multiply by 100 the Square of the Divisors of 24 and 40, and then find 32813, for the Measure of the Conoid, as found before.

Cor. 5. If the generating Curve be the Semi-Ellipsis VMN (*Fig.* 42.) about its transverse $Vv=2b$, and the Conjugate $MN=b$, and if this be cut by a Plane, parallel to the Base MON, which is distant from V by $Vo=x$, whose Section with MVN, is the Line $mn=y$, then the Measure of the

$$\textit{Segment, } Vmnm = \overline{\frac{b^2x}{b}+y^2}\times\frac{ax}{3} = \overline{3y^2+\frac{b^2x^2}{b^2}}\times\frac{ax}{6}$$

$$\textit{Semi-Spheroid } VMNM = 2b^2\times\frac{ab}{3}$$

$$\textit{Frustum } mnMN = \overline{2b^2+y^2}\times\frac{\overline{b-x}\times a}{3}$$

For (by *Prop.* 1. *Chap.* 2. *Part.* 1.) $b^2 : b^2 :: \overline{2b-x}\,xx : y^2 = \frac{2b^2x}{b} - \frac{b^2x^2}{b^2}$: this Compared with $y^2=Ax+Bx^2$ (the Equation of the Proposition) gives $A=\frac{2b^2}{b}$, $B=-\frac{b^2}{b^2}$, and therefore by *Cor.* 1. $S=\overline{\frac{b^2x}{b}+y^2}\times\frac{ax}{3}=Vmnm$ the Segm. In which making $x=b$ and $y=b$, we have $2b^2\times\frac{ab}{3}=VMNM$ the Semi-spheroid; and by (*Cor.* 2.) we have $F=\overline{\frac{2b^2}{b}\times\overline{b+x}-\frac{b^2x}{b}+2b^2+2y^2}\times\frac{\overline{b-x}\times a}{6}=\overline{2b^2+y^2}\times\frac{\overline{b-x}\times a}{3}$, the Frustum $mnMN$, and from hence the three following Rules for measuring these Solids by the Pen.

Rule

Rule 1. *To measure any Segment of a Spheroid.*

Let the Square of the conjugate Axis of the Spheroid be multiplied by the Height of the Segment, and call the Quotient thereof divided by the Semi-transverse a, to which add the Square of the Diameter of the Segment's Base; and multiply the Sum by the Height of the Segment, and that Product by ,2618, which will give the Measure of the Segment proposed.

Rule 2. *To measure any Spheroid.*

Multiply the Square of the conjugate Axis of the generating Ellipse by the transverse Axis thereof, and that Product by ,5236, so shall you have the Measure of the Spheroid proposed.

Rule 3. *To measure any Frustum of a Spheroid.*

To twice the Square of the Diameter of the greater End, add the Square of the Diameter of the lesser End; let this Sum be multiply'd by the Height or Length of the Frustum, and that Product by ,2618, which will give the Measure of the Frustum sought.

Examp. Let the transverse Axis be $40=2b$, the Conjugate $30=b$, the Height of the Segment $8=x$, then $b^2 : b^2 :: \overline{2b-x} \times x : y^2 = 576$, and $b-x=12$.

OPE-

Operation.

For the Segment.

$$
\begin{array}{lr}
b = \text{ - - - - - } & 30 \\
 & 30 \\
\hline
b^2 = \text{ - - - - } & 900 \\
x = \text{ - - - - - - } & 8 \\
\hline
b = \text{ - - } & 20)\ 7200\ (360 = \frac{b^2 x}{b} \\
\hline
\frac{b^2 x}{b} = \text{ - - - - } & 360 \\
y^2 = \text{ - - - } & 576 \\
\hline
\frac{b^2 x}{b} + y^2 = \text{ - } & 936 \\
 & 8 \\
\hline
\frac{b^2 x}{b} x + y^2 \times x = & 7488,0000 \\
.2618 \text{ inverted} & 21620 \\
\hline
 & 14976000 \\
 & 4492800 \\
 & 74880 \\
 & 59904 \\
\hline
mVn = \text{ - - } & 1960.3584
\end{array}
$$

OPERATION *for the Spheroid.*

$b =$ - - - -	30
	30
$b^2 =$ - - -	900
$2b =$ - - -	40
$b^2 \times 2b =$ - - -	36000
	,5236
	216000
	108
	72
	180
$b^2 \times 2b \times .5236 = \text{VMN} =$	18849,6000
From its half = -	9424,8000
Take the Segment =	1960,3584
The Remainder =	7464,4416

Is the Frustum as found by the next Operation.

OPERATION *for the Frustum.*

$b =$ - - - -	30
	30
$b^2 =$ - - -	900
	2
$2b^2 =$ - - -	1800
$y^2 =$ - - -	576
$\overline{2b^2+y^2} =$ - -	2376
$\overline{b-x} =$ - -	12
	4752
	2376
$\overline{2b^2+y^2} \times \overline{b-x} =$ -	28512.000
2618 inverted -	8162
	57024000
	17107200
	285120
	228096
mnMN $=$ - -	7464,4416

And from hence we have the three following Rules by the Sliding-Rule.

Rule 1. *To measure the Segment of a Spheroid by the Sliding-Rule.*

First set the Semitransverse on B to the Conjugate on A, and call the Number a, which is on A opposite the Height of the Segment on B; then set

set 2.764 on D to the Height of the Segment on C, and find both the Number α, and the Diameter of the Segment's Base on D, then if to the Number on C opposite α be added triple the Number opposite the said Diameter, that Sum will be the Measure of the Segment; so in the present Case, first I set 20 on B to thirty on A, then opposite 8 on B is 12 on A, which is α: then setting 2,764 on D to 8 on *c* against 1,2 (for 12 is off the Rule) is 1,508; and against 2,4 is 6,0318, which triple, is 18,0954; this added to 1,508 gives 19.6034, which multiplied by 100 (the Square of the Divisor of α and 24) gives 1960.34, nearly the same as was found before.

Rule 1. *To measure a Spheroid by the Sliding-Rule.*

Set 1.382 on D to the transverse Axis on C, then against the conjugate Axis on D is the Measure of the Spheroid on C.

Thus in the present case 1.382 on D being set to 40 on C against 3 on D (for 30 is off the Rule) is 188.5 on C; this multiplied by 100 (the Square of the Divisor of 30) gives 18850. for the Measure of the Spheroid, nearly the same as before.

Rule 3. *To measure the Frustum of a Spheroid by the Sliding Rule.*

Set 1.954 on D to the Height or Length of the Frustum on C, then the greater and lesser Diameters of the Frustum's Ends being found D; to twice the Number opposite the first on C, add that opposite the lesser on C, and this Sum will be the Measure of the Frustum sought.

So here, 1,954 on D being set to 12 on C, against 3 on D is 28.274 on C, and against 2.4 on D is 18.096 on C, to which add 56.548 the double

of 28,274, and you have 74.644; where removing the Decimal Point two Places towards the right Hand (because both 30 and 24 were divided by 10) we find 7464.4 for the Measure required; nearly the same as before.

Cor. 6. If in the Expression above for the Spheroid, its Segment and Frustum, you put $b = \frac{b}{2}$, you'll have the following Expressions for the Sphere, its Segment and Frustum, (*vide Fig.* 43.) where MN=*b*. the Measure of the

Segment V*mnn* is $\overline{2bx+y^2}\times\frac{ax}{3}=\overline{3y^2+4x^2}\times\frac{ax}{6}$.

Semisphere VMNM is $\frac{b^3\times a}{3}$.

Frustum *mn*MN is $\overline{2b^2+y^2}\times\overline{b-2x}\ \frac{xa}{6}$.

And these Expressions put into Words, give the three following Rules for measuring the Sphere, its Segment and Frustum by the Pen.

Rule 1. *To measure the Segment of a Sphere.*

To the Square of the Diameter of the Segment's Base, add twice the Product of the Diameter of the Sphere by the Height of the Segment, and multiply this Sum by the Height of the Segment; then this last Product multiplied by ,2618 gives the Measure of the Segment proposed.

Rule 2. *To Measure any Sphere.*

The Cube of the Sphere's Diameter multiplied by ,5236, gives the Measure thereof.

Rule 3. *To measure the Frustum of a Sphere.*

To twice the Square of the Sphere's Diameter, add the Square of the end; and multiply the Sum

by

by the Height of the Frustum, and that Product by ,2618 ; so will you have the Measure sought.

Examp. Let the Diameter of a Sphere be 25 Inches, the Height of a Segment thereof 9 Inches, and consequently the Diameter of its Base 24 Inches, and the Height of the Frustum 3.5 Inches.

OPERATION *for the Segment.*

$y =$ - - - -	24
	24
	96
	48
$y^2 =$ - - -	576
$2bx = 50 \times 9 =$ -	450
$2bx + y^2 =$ - -	1026
$x =$ - - -	9
$\overline{2bx + y^2} \times x =$	9234,0000
2618 inverted -	8162
	18468000
	5540400
	92340
	73872
$mVn =$ - - -	2417,4612

OPERATION *for the Sphere.*

b = - - -	25
	25
	125
	50
b^2 = - - -	625
	25
	3125
	1250
b^3 = - - -	15625,0000
.5236 inverted	6325
	78125000
	3125000
	468750
	93750
$b^3 \times ,5236$	=8181.2500

OPE-

OPERATION *for the Frustum.*

$b =$	25
	25
	125
	50
$b^2 =$	625
	2
$2b^2 =$	1250
$y^2 =$	576
$\overline{2b^2+y^2} =$	1826
$\frac{b-2x}{2} =$	3·5
	9130
	5478
$\overline{2b^2+y^2} \times \overline{b-2x} =$	6391,000
,2618 inverted	8162
	1278 2000
	383 4600
	63910
	51128
$\overline{2b^2+y^2} \times \frac{b-2x}{2} \times ,2618 =$	1673,1638

And from these Rules we may deduce three others, for measuring these Solids by the Sliding-Rule.

Rule 1. *To measure a spherical Segment by the Sliding-Rule.*

Set 2.764 on D to the Height of the Frustum on C, and find the Diameter of the Segment Base, and also the Height of the Segment on D; then to triple the Number on C opposite the first, add quadruple that opposite the second, so shall this Sum be the Measure required.

So in the above *Example*, 2.764 on D set to 9 on C, against 2.4 on D, is 6,7866 on C, and against 2.5 is 95.43; now because 2.5 was divided by 10, 5678,6 is the first Number, which tripled is 2035,8; and 95.43 quadrupled is 381.72, these added together, give 2417.52, the Measure sought, nearly the same as before.

Rule 2. *To measure a Sphere by the Sliding-Rule.*

Set 2.382 on D to the Diameter of the Sphere on C, then against the said Diameter on D, is the Measure sought on C.

In the preceding *Example*, I set 1,382 on D to 2,5 on C; then against 2.5 on D is 81,812, in which removing the decimal Point two Places forwards (because 2.5 is $\frac{1}{10}$ of 25) and we have 8181.2 for the Measure of the Sphere.

Rule 3. *To measure the Frustum of a Sphere by the Slyding-Rule.*

Set 1.954 on D to the Height of the Frustum on C, and find the Measures of the Diameters of the Ends on D, then twice the Number on C against the greater, added to that against the less, is the Measure required.

In the present Case 1,954 on D, set to 3.5 on C; then against 25 on D is 572.69, which doubled is

1145.38

1145,38, and against 24 is 527,79, which added to 1145,38 gives 1673,17 nearly the same as before.

Cor. 7. If the generating Curve be the Hyperbola PVQ (*Fig.* 39.) about the Axis VC=b, the Diameter PQ=b, and suppose this cut by a Plane parallel to the Base PQ, distant from the Vertex V by VO=x, the double Ordinate MN=y, and the transverse Axis of the Hyperbola equal to t; then the Measure of the Segment VMN $= \frac{3t+2x}{t+b} \times \frac{abyz}{6}$,

Frustum PQNM $= \overline{\frac{3t+2b}{t+b} \times bb^2 - \frac{3t+2x}{t+x} \times xy} \times \frac{a}{6}$.

For call the Parameter of the Hyperbola p, then (by *Prop.* 3. *Chap.* 2.) $t : p :: tx+x^2 : \frac{y^2}{4}$; therefore $y^2 = 4px + \frac{4px^2}{t}$. Let this Equation be compared with that in the Proposition, then A$=4p$, and therefore (by *Cor.* 1.) S $= \overline{4px+2y^2} \times \frac{ax}{6}$; and this when x becomes b, and consequently $y=b$ is PVQ$= \overline{4pb+2b^2} \times \frac{ab}{6} = \frac{3t+2b}{t+b} \times \frac{abb^2}{6}$; for the same reason MVN $= \frac{3t+2x}{t+x} \times \frac{axy^2}{6}$; and the Difference of these is the Measure of the Frustum PQMN$=$ $\overline{\frac{3t+2b}{t+b} - xy^2 \times \frac{3t+2x}{t+x}} \times \frac{a}{6}$. From hence it is manifest to measure the hyperbolic Conoid, besides the Data that was required in the preceding ones, we must know the Measure of the transverse Axis of the generating Hyperbola; and to measure the Frustum, the whole Height of the Conoid must be found from the Equation of the Curve; that is the *Sphere*, *Spheroid* and *parabo-*

lic

lic Conoid are in a determinate Ratio to their *circumscribing Cylinders*, but in the hyberbolic Conoid, that Ratio is variable.

We ſhall add no Example to this, becauſe it ſeldom can occur in Gauging; and from what has been ſaid above, it will be eaſy for the Reader to ſupply that of himſelf.

Cor. 8. If the generating Curve be a Semi-Ellipſe VNv (*Fig.* 42.) about its *Semiconjugate* Axis NO, which now let be b, and its Tranſverſe b, thereby generating the *Semioblate Spheroid* VNv, which is cut by a Plane nn' parallel to Vv, at the Point o, in the revolving Axis, diſtance from N the Vertex, by N$o = x$; and the correſponding Ordinate to this, *viz.* nn' be put equal to y, then the Meaſure of the Segment nNn' is

$$\overline{\frac{b^2x}{b}+y^2}\times\frac{ax}{3}=\overline{\frac{b^2x^2}{b^2}+3y^2}\times\frac{ax}{6};$$

The Semioblate Spheroid VNv is $\frac{2b^2\times ab}{3}$;

The Fruſt. $nn'v$V is $\overline{2b^2+y^2}\times\overline{b-x}\times\frac{a}{3}$.

Which are analogous to thoſe for the common Spheroid, its Segment, and Fruſtum *Cor.* 5. foregoing; for here the Equation of the generating Curve is $y^2=\frac{2b^2x}{b}-\frac{b^2}{b^2}x^2$; whence for A $=\frac{2b^2}{b}$, and B $=\frac{b^2}{b^2}$, (*Cor.* 1 and 2, foregoing) theſe Values being put, you'll have the Expreſſions above.

Having in the preceding Propoſition, and its Corollaries, given the Meaſure of the Cone, Conoid, their Fruſtums and Segments, we ſhall now ſhew how to find the Meaſure of a *ſecond Segment*, or Slice cut off by a Plane, parallel to the Axe of Revolution; in order to which, we muſt premiſe the following Lemmas. LEMMA

LEMMA I.

If a Parabolic Conoid be cut by a Plane, parallel to the Axis of Revolution, the Figure of the Section will also be a Parabola, and the Parameter thereof same with that of the Generating Parabola.

For let MVN (*Fig.* 44.) be the Conoid, and *mvn* the Section thereof, by a Plane parallel to VO the Axis of Revolution, which cuts the generating Parabola, in the Line *vo*; then will *mnv* be a Parabola, whose Parameter is equal to that of MVN; call it *p*, then (by *Cor.* 3. *Prop.* 2. *Chap* 2.) $p \times vo = Mo \times oN$, but $Mo \times oN = \overline{on}|^2$, (8 *Euclid* 6.) therefore $p \times vo = \overline{on}|^2$, which is the Property of a Parabola, whose Parameter is *p*, Abscissa *vo*, and ordinate *on*; therefore, *&c.*

LEMMA II.

If a Spheroid be cut by a Plane, parallel to the Transverse or Axis of Revolution, the Figure of the Section will also be an Ellipsis, the Transverse and Conjugate of which will be to those of the generating Ellipse in a direct Proportion.

For let VM*v*RM (*Fig.* 46.) be the Semispheroid, generated by the Semi-Ellipse VR*v* about its Transverse Axis V*v*, whose Center is Q, and ED its Section by a Plane parallel thereto, so that E*mn*D*mn* is the Section of that Plane and the Spheroid: Let two Planes be drawn perpendicular to the Transverse, one of which passes thro' the Center Q, making the Semicircle MRM, and the other thro' any Point *q* (in V*v*), making the Circle N*r*N, and cutting the Plane E*mn*D*mn*E, in the Lines *mm*, *nn*; I say E*mn*D*nm*E is an Ellipsis, and ED its Transverse, is to *mm* its Conjugate, as V*v* is to 2OR.

For

For put VQ=T, QR=C, OD=t, Om=c, Oo=x=on=y; then from the Property of the Circle $\overline{rq+qo}\times\overline{rq-qo}=\overline{rq}^2-\overline{oq}^2=\overline{on}^2=y^2$; but (by *Cor.* 2. *Prop.* 1. *Chap.* 2.) $\overline{rq}^2=C^2-\frac{C^2x^2}{T^2}$, and for the same Reason $\overline{oq}^2=\overline{EF}^2=C^2-\frac{C^2t^2}{T^2}$; therefore $y^2=\frac{C^2}{T^2}\times\overline{t^2-x^2}$, where making $x=o$ then $y=c$, but then $c^2=y^2=\frac{C^2t^2}{T^2}$, or $c=\frac{Ct}{T}$, that is $T:C::t:c$, whence $y^2=\frac{c^2}{t^2}\times\overline{t^2-x^2}$, which (by *Cor.* 2. *Proposit.* 1. *Chap.* 2.) is the Equation of an Ellipse, whose Transverse is $2t$, and Conjugate $2c$.

PROP. III.

If the Parabolic Conoid MVN (Fig. 44.) *whose Axis is* VO=h *the Diameter of its Base* b, *be cut by a Plane* mvn *parallel to* VO, *and distant from it by* oO=z, *the Measure of the Slice* mnvM *shall be expressed by this*

THEOREM, *viz.*

$$mnvM=\overline{\frac{S}{2}-\frac{z\times\overline{b^2-4z^2}\sqrt{b^2-4z^2}}{6b^2}}\times b,$$ where S denotes the Measure of the circular Segment mMn, *viz.* of a Circle, whose Diameter is b, and versed Sine $\frac{b-2z}{2}$.

For put $mn=y=\sqrt{b^2-4z^2}$, then because by the first Lemma above, mvn is a Parabola, whose Para-

Parameter is $\frac{b^2}{4b}$, we have (by *Prop.* 2. *ch.* 2. *part* 2.) $\frac{b^2}{4b} \times vo = \frac{y^2}{4}$: but $\frac{2vo \times mn}{3}$ is the Measure of *mvn*, (by *Prop.* 4. *Chap.* 3.) therefore $\frac{2by^2}{b^2} \times \frac{y}{3} = \frac{2by^3}{3b^2}$ is equal to *mvn*.

Let this be drawn into $\frac{y\dot{y}}{2\sqrt{b^2-y^2}}$ the Fluxion of $Mv = \frac{b-\sqrt{b^2-y^2}}{2}$, and we have $\frac{b}{3b^2} \times \frac{y^4\dot{y}}{\sqrt{b^2-y^2}}$ for the Fluxion of the Measure sought; but $\frac{y^2\dot{y}}{2\sqrt{b-^2y^2}}$ $=$S. Now from the 7th *Proposition* of the Book of Quadratures, if P be the Area of a Curve, whose Ordinate is $py^{\theta-1} \times \overline{e+fx^n}|^{\lambda-1}$, and Q the Area of another Curve, whose Ordinate is $q^{\theta+n-1} \times \overline{e+fy^n}|^{\lambda-1}$, then $Q = \frac{q}{p} \times \frac{py^\theta \times \overline{e+fy^n}|^\lambda - e\theta P}{f \times \lambda n+\theta}$: therefore because here $p = \frac{1}{2}$, $q = \frac{b}{rb^2}$, $\theta = 3$, $\lambda = \frac{r}{2}$, $e = b^2$, $f = -1$, and $n = 2$, we have $Q = \overline{\frac{P}{2} - y^3 \frac{\sqrt{b^2-y^2}}{12b^2}} \times b$, in which for P write S, and for y, b^2-z^2, then $\frac{S}{2} - \overline{\frac{z \times b^2-4z^2 \sqrt{b^2-4z^2} \times b}{6b^2}}$ is the Measure of the Slice *mnv*M. *Q. E. O.*

Cor. 1. Let M*n*N*m* (*Fig.* 45.) be equal to the Base of the Conoid, MN its Diameter, and M*o* $=$M*o*; thro' *o* draw the Perpendicular *mon*, meeting the Circle in the Points *m*, *n*, from *m*. Thro' the Center O draw the Line *m*O*k*, and from *n*, let fall a Perpendicular thereon, meeting the same in

in l, then the Solidity of the Slice mnvM of the Conoid (*Fig.* 44.) may be exprest by $\overline{\frac{mn\text{M}}{2} - \frac{mnl}{6}} \times b$, for (*per Figure*) $b : y :: y : \frac{y^2}{b} = ml$, and $b : \sqrt{b^2 - y^2} :: y : \frac{y\sqrt{b^2 - y^2}}{b} = nl$; therefore $\frac{y^2}{2b} \times \frac{y\sqrt{b^2 - y^2}}{b} = y^3 \frac{\sqrt{b^2 - y^2}}{2b^2}$ is the Measure of the Triangle mnl; consequently $\overline{\frac{mn\text{M}}{2} - \frac{mnl}{6}} \times b$ is equal to $\overline{\frac{\text{P}}{2} - \frac{y^3\sqrt{b^2 - y^2}}{12b^2}} \times b = mnv$M.

Cor. 2. If nl (*Fig.* 45.) be continued till it meet the Circumference in w, and the Line mw drawn, then the Segment nM$mw = 2mn$M$+2nml$, whence $\overline{8mn\text{M} - n\text{M}mv} \; (= \overline{6mn\text{M} - 2mnl} \times \frac{b}{12} = \overline{\frac{mn\text{M}}{2} - \frac{mnl}{6}} \times b)$ is the Measure of mnvM, which is the Theorem publish'd by Mr. *Hunt*, and said to be received from Sir *Isaac Newton*; but what we have given above is more simple, it requiring but the Area of one Segment of a Circle to be found, whereas by this Method you must have the Area of two.

For an Example, Let MN=92 Inches, Mo=31 Inches, and VO=294 Inches, then by the Theorem (*Pag.* 100.) I find the Measure of the Segment mMn equal to 1968 nearly, and because Mo=31, z=15, therefore $\frac{z \times \overline{b^2 - 4z^2} \times \sqrt{b^2 - 4z^2}}{6b^2} = \frac{15 \times 7564 \times \sqrt{7564}}{6 \times 84164} = 194.31$ *proxime*; hence according to the Theorem $\frac{1968}{2} - 19431 = 789.69 \times$ 294=

294=232169 nearly, the Meaſure of the Slice *mn*V*m*. *Q. I. E.*

In order to ſhew the Invention of the above Theorem, I was obliged to uſe the Method of Fluxions, becauſe beſides the *Lemma*, *Pag.* 85, I muſt have inſerted ſeveral more (too tedious for this Place) in order to render a Demonſtration but tolerably clear.

Prop. IV.

If a Spheroid generated from the Ellipſe VQvR, *whoſe tranſverſe Axis is* Vv=t (Fig. 46.) *and Semiconjugate* QR $= \frac{c}{2}$ *be cut by a Plane thro'* O *in* QR, *diſtant from* R, *the Vertex of the Conjugate by* RO=*x*, *and* ED *the Section of the generating Ellipſe be called* y, *the Meaſure of the Segment of the Spheroid* EDR *ſhall be expreſſed by this*

Theorem, *viz.* EDR$=\overline{2t^2x+cy^2}\times\frac{ax}{3t}$.

For conceive OR divided into an Infinite Number of Parts, the firſt Diviſion of which from O is at O′, and let q denote the Meaſure of the ſmall Part OO′, now 'tis manifeſt (by *Lemma* 2. foregoing) E*m*D*m*E, the Section of the Cutting-Plane with the Spheroid, is an Ellipſe ſimilar to the generating Ellipſe VRv; therefore $t:c::y:\frac{cy}{t}=mm$ the Conjugate of the Section E*m*D*m*E, but becauſe MRM (ſee *Lemma* 2.) is a Circle whoſe Diameter is c, we have $mm=2\sqrt{cx-x^2}$, therefore $\frac{cy}{t}=2\sqrt{cx-x^2}$; but all ſimilar Figures are as the Square of their homologous Sides, (as is demonſtrated further on) and the Area of the generating

rating Ellipse is $axtxc$, (by *Cor.* 1. *Pr.* 5. *Ch.* 3.) therefore $c^2 : axtxc :: 4x\overline{cx-xx} : \frac{4at}{c} \times \overline{cx-x^2}$ the Measure of the Ellipse EmDmE, and this drawn into p, gives the Measure of the first or greatest Elementa of the Segment sought, call it e; and the succeeding ones e', e'', e''', &c. the corresponding Values of RO, x', x'', x''', &c. Then we have the Measure of the several Elementa, as here under.

$$e = \frac{4qat}{c} \times \overline{cx-x^2}$$

$$e' = \frac{4qat}{c} \times \overline{cx'-x'^2}$$

$$e'' = \frac{4qat}{c} \times \overline{cx''-x''^2}$$

$$e''' = \frac{4qat}{c} \times \overline{cx'''-x'''^2}$$

and the Sum of these is the Measure of the whole Segment sought, but the Number of the Elementa is $\frac{x}{q}$, and the Quantities x, x', x'', x''', &c. are in arithmetic Progression; hence (by the *Lemma* P. 85.) $\frac{4qat}{c} \times \frac{cx^2}{2q} - \frac{4qat}{c} \times \frac{x^3}{3q} = \frac{4at}{c} \times \overline{\frac{cx^2}{2} - \frac{x^3}{3}}$, is the Measure of RD$mEm$ $= \frac{2atx}{3c} \times \overline{3cx-2x^2}$: In which for x put its Value $cx - \frac{c^2y^2}{4t^2}$, and we have $\overline{2t^2x+cy^2} \times \frac{ax}{3t} =$ RDmEm. *Q. E. O.*

Corol. 1. Therefore making $x = \frac{c}{2}$ and $y = t$, then for the Measure of half the Spheroid we have

$2t^2 \times \frac{c}{2} + ct^2 \times \frac{ac}{6t} = \frac{a \times tc^2}{3}$, the ſame as was ſhewn before, *Cor.* 5. *Prop.* 2.

Cor. 2. If from the Meaſure of the Semiſpheroid $\frac{a \times tc^2}{3}$, that of the Segment, *viz.* $\overline{2t^2x + cy^2} \times \frac{ax}{3t}$, be taken, we ſhall have for the Meaſure of the Fruſtum this

THEOREM, *viz.* $VvDE = \overline{2t^2 + y^2} \times \frac{axc}{3t} \times \frac{c - 2x}{2}$.

And the firſt of theſe Theorems being turn'd into Words, give the following *Rule for meaſuring a parallel Segment of a Spheroid.*

RULE.

Multiply the Square of the Tranſverſe Axis of the generating Ellipſe by twice the Height of the Segment; and the Square of the Tranſverſe of the Section of the Spheroid, by the Conjugate of the generating Ellipſe, and add theſe two Products together, then multiply their Sum by the Product of ,2618 by the Height of the Segment; this laſt Product divided by the tranſverſe Axis of the generating Ellipſe, is the Meaſure of the Segment ſought.

Examp. Let the Tranſverſe of the generating Ellipſe be 73 Inches, its Conjugate 29 Inches, the Height of the Segment 2 Inches, and therefore the Tranſverſe of the Section 25 Inches, to find the Meaſure of the Segment RD*m*E*m*.

OPERATION.

$y =$ - 37   73 $= t$
37   73

259   219
111   511

$y^2 =$ - 1369   5329 $= t^2$
$c =$ - 29   4 $= 2x$

12321   21316 $= 2t^2x$
2738

$cy^2 =$ 39701 $=$ 39701 $= cy^2$

61017 $= 2t^2x + cy^2$
2 $= x$

$3t = 219)$ 122034 | 557,20 $= \overline{2t^2x + cy^2} \times \frac{x}{3t}$
1095 | 4587,0

1253 | 39004
1095 | 4458
| 279
1584 | 22
1533 |

51 | 436,63 $= \overline{2t^2x + cy^2} \times \frac{ax}{3t}$

Corol. 3. But if the above Segment should be cut by a Plane through some Point *r*, and parallel to the Conjugate RO, then the Piece cut off is called a *Second Segment* of the Spheroid; to measure which by a finite Expression, is impossible, even tho' you use the Segment of a Circle; at least

leaſt by any Method I yet have met with: but when oD is ſmall in reſpect to OD, it may be had by an Approximation, ſufficiently near for Practice, from what has been ſaid in the laſt Propoſition: for in the Figure put $rq = \frac{b}{2}$, $qo = z$, $oD = l$, and denote the circular Segment *nrnon* by S, then the Meaſure of the ſecond Segment *Drnn* is nearly had by this

THEOREM, *viz.* $Drnn = \overline{\frac{3b^2 \times S}{b^2 - 4z^2} - z\sqrt{b^2 - 4z^2}} \times \frac{l}{6}$.

Corol. 4. From hence, and *Prop.* 3. we may compute the Meaſure of the Part *mRmnnr* (*Fig.* 46.) for by changing the Form of that Theorem, *viz.* for x put its equal $\frac{c - 2z}{2}$, from this Suppoſition of $QO = qo = z$, then $\frac{t}{c} = \frac{2L}{\sqrt{c^2 - b^2}}$, and $l = \frac{t}{2c}\sqrt{c^2 - 4z^2} - \frac{L}{2}$: ſo that the Meaſure of the ſecond Segment $OoRr$, which is nearly $\overline{c + z \times c - 2z}^2 \times \frac{ta}{rc} + \overline{z\sqrt{b^2 - 4z^2} - \frac{3b^2 \times S}{b^2 - 4z^2}} \times \frac{l}{6}$, may be computed from the Meaſures of RQ, rq, Oo, and OQ ($= oq$) being given; that is the Bung and Head-Diameters of a Spheroidal Caſk being given, and alſo its Length, and the Diſtance of the Surface of the Liquor from the Axis thereof, we may by this *Corol.* nearly compute how much is drawn off: But Note, if the Caſk be more than half empty, you will then find how much remains in it; which taken from the whole Content, leaves what is drawn off. But this Theorem requires too

much labour to be followed by the Officers of Excise, so we shall give other Methods for that purpose further on.

But I must observe this is exceeding near the Truth, especially if for S you put $\frac{121b+78z}{29b+6z} \times \frac{\overline{b-2z}}{15} \times \sqrt{2b \times \overline{b-2z}}$, which is deduced from that *Page* 101. by putting $\frac{b-2z}{2}$ for v.

We might also give a like Approximation for the *Second Segments* of the Hyperbolic Conoids, for they as well as those of the Spheroid, do not admit of a perfect Cubation; but as no Casks are ever made resembling this Solid, I shall not detain the Reader with useless Speculations, but proceed to shew how to measure the Spindles, which are generated by the Revolution of a Conic Section round an Ordinate to the Axis.

Prop. V.

Let AVBvA (Fig. 47.) *be a Spindle, generated by the Parabola* AVB*'s turning round its double Ordinate* AB; *and thro'* n *a Point in the Axe of Revolution, let a Plane be imagined to pass perpendicular to it, whose Section with the Spindle is the Circle* Mnm; *(put* Vv=b, Mm=y *and* Cn=x*) then the Measure of the Frustum* VMmv *shall be expressed by this*

Theorem, *viz.*

$$VMmv = \overline{2b^2 + y^2 - \tfrac{2}{5}\cdot\overline{b-y}|^2} \times \frac{\text{[illegible]}}{3}.$$

For call the *Parameter* of the generating Parabola p, and suppose Cn divided into an infinite Number of Parts, each of which is measured by q; and denote the successive Values of x from n

to

to C by x', x'', x''', *&c. viz.* $x'=x-q$, $x''=x'-q$, $x'''=x''-q$, *&c.* alſo let e, e', e'', e''', *&c.* denote the correſponding Elementa of the Sol.d.

Now (by *Prop.* 2. *Chap.* 2.) $p\times VO=\overline{MO}|^2$, that is $p\times\frac{b-y}{2}=x^2$, therefore $y=b-\frac{2x^2}{p}$, and $y^2=b^2-\frac{4bx^2}{p}+\frac{4x^4}{p^2}$; whence the Value or Meaſure of of the firſt Elementa (e) is $aq\times\overline{b^2-\frac{4bx^2}{p}+\frac{4x^4}{p^2}}$; and from hence the following Equations;

$$e=aq\times\overline{b^2-\frac{4bx^2}{p}+\frac{4x^4}{p^2}}$$

$$e'=aq\times\overline{b^2-\frac{4bx'^2}{p}+\frac{4x'^4}{p^2}}$$

$$e''=aq\times\overline{b^2-\frac{4bx''^2}{p}+\frac{4x''^4}{p^2}}$$

$$e'''=aq\times\overline{b^2-\frac{4bx'''^2}{p}+\frac{4x'''^4}{p^2}}$$

&c. *&c.*

And ſince q *Diſtance* gives 1 Term, x *Diſtance* will give $\frac{x}{q}$ *Number of Terms*; that is $\frac{x}{q}$ is the Number of Elementa, or Terms of the above Series, each of which is alſo compoſed of three Terms; the Sum of all the *firſt* is $aq\times b^2\times\frac{x}{q}$ $=ab^2\times x$, and (by the *Lemma*, *Pag.* 85.) the Sum of all the *ſecond* Terms, *viz.* $aq\times\frac{4b}{p}\times$ $\overline{x^2+x'^2+x''^2}$, *&c.* is $aq\times\frac{4b}{p}\times\frac{x^2}{3}\times\frac{x}{q}=\frac{4abx^3}{3p}$,

 and

and (by the ſame *Lemma*) the Sum of all the *third* Terms, *viz.* $aq \times \frac{4}{p^2} \times \overline{x^4 + x'^4 + x''^4}$, &c. is $aq \times \frac{4}{p^2} \times \frac{x^4}{5} \times \frac{x}{q} = \frac{4ax^5}{5p^2}$, whence collecting theſe Sums under their reſpective Signs, we have VM*mv* $= \overline{b^2 - \frac{4bx^3}{3p} + \frac{4x^5}{5p^2}} \times ax$: but $\frac{2x^2}{p} = b - y$, therefore the laſt Theorem is equivalent to $\overline{b^2 - \frac{2b \times \overline{b-y}^2}{3} + \frac{\overline{b-y}^4}{5}} \times ax$, that is to $\frac{\overline{8b^2 + 4by + 3y^2}}{5} \times \frac{a}{3}$, and this to $\frac{10b^2 + 5y^2 - 2b^2 - 2y^2 + 4by}{5} \times \frac{ax}{3} = \overline{2b^2 + y^2 - \frac{2}{5} . \overline{b-y}^2} \times \frac{ax}{3}$. *Q. E. O.*

Cor. 1. Whence if $y = o$, then the Meaſure of the whole Spindle is $\overline{2b^2 - \frac{2}{5}b^2} \times \frac{a \times AB}{3}$, or $\frac{8}{15} \times b^2 a \times AB$, that is $\frac{8}{15}$ths of its circumſcribing Cylinder.

From this Propoſition we have the following Rules.

Rule 1. *To Meaſure the Fruſtum of a parabolic Spindle.*

To twice the Square of the Diameter of the greater End, add that of the leſſer, and from the Sum take $\frac{2}{5}$ths, or (,4=)$\frac{4}{10}$ths of the Square of their Difference, and multiply the Remainder by the Height of the Fruſtum; then if this Product be multiplied by ,2617994=,2618 nearly, it will give the Meaſure ſought.

Rule 2. *By the Sliding-Rule.*

Set 1,954 on D, to the Height of the Fruſtum on C; then find the greater and leſſer Diameters, as

as alſo their Difference on D, noting the three Numbers oppoſite them on C; thèn if the ſecond is added to twice the firſt, and $\frac{2}{5}$ths, or $\frac{4}{10}$ths of the third taken from the Sum, you'll have the Meaſure requir'd.

Examp. Let (b) the greater Diameter be 32 Inches, (y) the leſſer 24, and (x) the Length 40 Inches.

OPERATION.

	32	$=b$
	32	
$y=24$	64	
24	96	
96	1024	$=b^2$
48		
	2048	$=2b^2$
576 =	576	$=y^2$
	2624	$=2b^2+y^2$
,4×8.8=25,6=	25,6	$=\frac{2}{5}\,.\,\overline{b-y}\vert^2$
	2598,4	$=2b^2+y^2-\frac{2}{5}\,.\,\overline{b-y}\vert^2$
	40	$=x$
	103936,0	
	8162,0	=,2618 inverted.
	20787 2	
	6236 1	
	103 9	
	83 1	
	27210,4	$=\overline{2b^2+y^2-\frac{2}{5}\,.\,\overline{b-y}\vert^2}\times\frac{ax}{3}$

 But

But by taking $a=2617994$ instead of ,2618, we should find the Measure to be 27210,3814 true to the last Figure.

By the Sliding-Rule. When 1.954 on D is set to 40 on C, then against 32, 24 and 8 on D, is 10723,3, 6031,9 and 670,2 respectively; therefore,

Twice the First	- - =	21446,6
More by the Second	- =	6031,9
		27478,5
Less by $\frac{4}{10}$ths the Third	=	268,08
Gives the Content	- =	27210,42

Prop. VI.

Let AVBvA (Fig. 48.) *be a Spindle generated by the elliptic Space* AVB *round* AB, *which is parallel to* TT, *the transverse Axis thereof; and thro'* n *a Point in* AB, *distant from* C, *the Middle of the Spindle by* x, *let a Plane be drawn perpendicular thereto, whose Section, with the generated Solid, is the Circle* Mnm: *and put* d=OC, *the Distance of* AB, *from* TT, *and* S *for the Measure of the elliptic Space* MVo; *then the Measure of* VMmvV, *a Frustum of the Solid, shall be expressed by this*

Theorem, *viz.*

$$\mathrm{VM}mv\mathrm{V}=\overline{2b^2+y^2+8d\times b-y-\frac{3S}{x}}\times\frac{ax}{3}.$$

For put t for the Transverse, and c the conjugate Axis of the generating Ellipsis, the rest as in the preceding Proposition, then (by *Prop.* 1. *Chap.* 2.) $t^2 : c^2 :: t^2-4x^2 : \overline{2d+y}|^2=\overline{\mathrm{M}m}|^2$, therefore $y=\frac{c}{t}\sqrt{t^2-4x^2}-2d$, and consequently $y^2\times q\times a$, the

the Measure of the first Elementa is, $qa \times \overline{c^2 + 4d^2 - \frac{4c^2x^2}{t^2} - \frac{4dc}{t}\sqrt{t^2 - 4x^2}}$; and from hence the following Equations.

$$e = qa \times \overline{c^2 + 4d^2 - \frac{4c^2x^2}{t^2} - \frac{4dc}{t}\sqrt{t^2 - 4x^2}}$$

$$e' = qa \times \overline{c^2 + 4d^2 - \frac{4c^2x'^2}{t^2} - \frac{4dc}{t}\sqrt{t^2 - 4x'^2}}$$

$$e'' = qa \times \overline{c^2 + 4d^2 - \frac{4c^2x''^2}{t^2} - \frac{4dc}{t}\sqrt{t^2 - 4x''^2}}$$

$$e''' = qa \times \overline{c^2 + 4d^2 - \frac{4c^2x'''^2}{t^2} - \frac{4dc}{t}\sqrt{t^2 - 4x'''^2}}$$

&c. &c.

Now (by the *Lemma*, *Pag.* 85.) 'tis manifest, the Sum of all the three first Terms of these Elementa, is $a \times \overline{c^2x + 4d^2x - \frac{4c^2x^3}{3t^2}}$, and the Sum of the fourth Terms are evidently equal to $8da \times$ MVON.

But the Measure of MVON is $S + x \times \overline{d + \frac{y}{2}}$, which being put for MVON, and $c^2 - \overline{2d + y}|^2$ for its Equal $\frac{4c^2x^2}{t^2}$, then we shall have the Measure of $VMmv = \overline{2c^2 + \overline{2d + y}|^2 - 12d^2 - 12dy} \times \frac{ax}{3} - 8dxS$; but $d + \frac{b}{2} = \frac{c}{2}$, therefore $2c^2 = 8d^2 + 8db + 2b^2$; put this for $2c^2$ in the last Expression, then for the Measure of $VMmv$ we shall find this

THEOREM,

THEOREM, *viz.*

$$VMmv = \overline{2b^2 + y^2 + 8d \times b - y} - \frac{3S}{x} \times \frac{ax}{3}.\ Q.E.O.$$

Cor. 1. If $t=c$, or the Ellipse in this Proposition should be chang'd into a Circle, then the elliptic Space VM*o* becomes the Circular Segment VM*o*, (*Fig.* 49.) and therefore for *Measuring of the Frustum of a circular Spindle*, we have this

THEOREM.

$$VMmv\ (Fig.\ 49.) = \overline{2b^2 + y^2 + 8d \times b - y} - \frac{3S}{x} \times \frac{ax}{3}.$$

Cor. 2. If (in the Theorem for measuring an Elliptick Spindle) for d be wrote $-d$, because O, *Fig.* 48. is on the contrary side C, to what it is *Fig.* 51. *Then for measuring the Frustum of an Hyperbolic Spindle we have this*

THEOREM.

$$VMmv\ (Fig.\ 50.) = \overline{2b^2 + y^2 - 8d \times b - y} - \frac{3S}{x} \times \frac{ax}{3},$$

where $b=$Vv, $y=$Mm, $x=$Cn, $d=$CO and S the Measure of the hyperbolic Space VM*o*.

But if any are not satisfied with these Inferences, they may by the same Steps with those for the Elliptic Spindle, investigate the two last Theorems *à priori*.

Cor. 3. If t be the transverse, and c the conjugate Axis of the Ellipse or Hyperbola, then for the

Ellipse

$$d = \begin{cases} \frac{t-b}{2} = \frac{t\times\overline{b-y}\times\overline{t+\sqrt{t^2-4x^2}}}{8x^2} + \frac{b}{2} & \textit{Ellipse} \\ \frac{t+b}{2} = \frac{c\times\overline{b-y}\times\overline{c+\sqrt{c^2+4x^2}}}{8x^2} + \frac{b}{2} & \textit{Hyperbola} \\ \frac{x^2}{b-y} - \frac{1}{4}\times\overline{b+y} & \textit{Circle} \\ \frac{x^2}{b-y} + \frac{1}{4}\times\overline{b+y} & \textit{Equil. Hyp.} \end{cases}$$

But if l be put to denote the Line nn, or $l=2x$, that is, if l denote the Length of a Cask, generated by a conic Section, round an Ordinate to the Axis, whose Bung-Diameter is b, Head b, and Q denote that Part of the generating Curve, whose Abscissa is $\frac{b-b}{2}$, and Base or double Ordinate l, then the Measure of these Casks will be found by the following Theorems, *viz.*

The

The $\left\{\begin{array}{l}\textit{Elliptic}\\ \textit{Circular}\\ \textit{Hyperbolic}\\ \textit{Equil. Hyp.}\end{array}\right\}$ *Spindle* $\left\{\begin{array}{l}\overline{2b^2+b^2+8d\times\overline{b-b}-\frac{3Q}{l}}\times\frac{al}{3}\\ =\\ \overline{2b^2+b^2-8d\times\overline{b-b}-\frac{3Q}{l}}\times\frac{al}{3}\end{array}\right.$

And the correſponding Values of d for theſe Solids are as follows, *viz.* For the

$$\left.\begin{array}{l}\textit{First}\\ \textit{Second}\\ \textit{Third}\\ \textit{Fourth}\end{array}\right\}d=\left\{\begin{array}{l}\frac{c-b}{2}=\frac{c\times\overline{b-b}\times\overline{c+\sqrt{c^2-l^2}}}{2l^2}-\frac{b}{2}\\ \frac{1}{4}\times\frac{l^2}{b-b}-\frac{1}{4}\times\overline{b+b}\\ \frac{c+b}{2}=\frac{c\times\overline{b-b}\times\overline{c+\sqrt{c^2+l^2}}}{2l^2}-\frac{b}{2}\\ \frac{1}{4}\times\frac{l^2}{b-b}+\frac{1}{4}\times\overline{b+b}\end{array}\right.$$

Cor. 4. By making $d=o$, we find the ſame Theorems for the Fruſtums of a Sphere and Spheroid, that was given at *Cor.* 5, 6. of the ſecond Propoſition.

And hence it alſo follows, if the Hyperbola MVM (*Fig.* 50.) revolve round NN, a Line paſſing thro' the Center of the Hyperbola, the Meaſure of the generated Solid MMM′M′ is $2\times\overline{Vv}|^2+\overline{MM}|^2\times\frac{a\times NN}{3}$, which is an Expreſſion ſimilar to thoſe

those for measuring the Frustum of the Sphere and Spheroid.

Cor. 5. By help of the Theorems for *Approximating* the circular and hyperbolic Segments, we might give the Measures of these Solids without Q, *viz.* expressed in the Terms of the Question: but as only the Circular, and Equilateral hyperbolic Spindles, seem to be of any use in Gauging, we have omitted the others, and only put down the Approximations for them, which are thus, *viz.* the

$$\left.\begin{array}{l}\textit{Circular}\\ \textit{Equil. Hyp.}\end{array}\right\}\textit{Spindle} = \left\{\begin{array}{l}\overline{2b^2+\beta^2-\dfrac{14\times\overline{b-\beta}^2\times\overline{l^2+b^2-\beta^2}}{35l^2+5\times\overline{b-\beta}^2}}\times\dfrac{al}{3}\\[2ex] \overline{2b^2+\beta^2-\dfrac{14\times\overline{b-\beta}^2\times\overline{l^2+b^2-\beta^2}}{35l^2-5\times\overline{b-\beta}^2}}\times\dfrac{al}{3}\end{array}\right\}\textit{nearly.}$$

For

For an Example to these two Cases; Let $b=32$, $b=24$, and $l=40$ Inches, then for the

$$\left.\begin{array}{l}\textit{Circular}\\ \textit{Eq.Hyp.}\end{array}\right\}\begin{array}{l}\textit{Spind.}\\ d=\end{array}\left\{\begin{array}{l}\frac{1}{4}\times\frac{1600}{8}-\frac{1}{4}\times 56\\ \frac{1}{4}\times\frac{1600}{8}+\frac{1}{4}\times 56\end{array}\right\}=\left\{\begin{array}{l}50-14=36\\ 50+14=64\end{array}\right.$$

Twice the first added to b, or 32, gives 104 for the Diameter of the generating Circle; and b, or 32, taken from twice the second, leaves 96 for the transverse Axis of the generating Hyperbola; whence from the Theorems, *Prop.* 3, and 6, of *Chap.* 2. For the

$$\left.\begin{array}{l}\textit{Circle}\\ \\ \\ \textit{Hyperb.}\end{array}\right\}Q=\left\{\begin{array}{l}\left\{\begin{array}{l}\frac{320}{3}+\frac{1}{5}\times\frac{1}{25}A-\frac{1}{7}\times\frac{1}{25}B+\frac{3}{9}\times\frac{1}{25}\\ \quad C-\frac{5}{11}\times\frac{1}{25}D+\&c.\end{array}\right.\\ \left\{\begin{array}{l}\frac{320}{3}-\frac{1}{5}\times\frac{1}{25}A-\frac{1}{7}\times\frac{1}{25}B-\frac{3}{9}\times\frac{1}{25}\\ \quad C-\frac{5}{11}\times\frac{1}{25}D-\&c.\end{array}\right.\end{array}\right.$$

$$\begin{array}{lr}\text{that is, } \frac{320}{3} = A = & 106.666667\\ \hline \frac{1}{5}\times\frac{1}{25}A = B = & 853333\\ \frac{1}{7}\times\frac{1}{25}B = C = & 4876\\ \frac{3}{9}\times\frac{1}{25}C = D = & 65\\ \frac{5}{11}\times\frac{1}{25}D = E = & 1\\ \hline B+C+D+E = & -,858275\\ \hline \end{array}$$

$A-B-C-D$, &c. $=105,808392=$ MVM *Fig.* 50. the Area of the Hyperbolic Segment.

$$\begin{array}{rr|rl} A = & 106,666667 & & \\ B = & 853333 & ,004876 = & C\\ D = & 65 & 1 = & E\\ \hline A+B+D = & 107,520065 & ,004877 = & C+D\\ & 004877 & & \\ \hline & 107,515188 & = & A+B-C+D-E \end{array}$$

The Area of the Circular Segment MVM *Fig.* 49.

Whence

Whence by the above Theorems, we have for the *Circular Spindle* this

OPERATION.

$$\begin{array}{r} 32=b \\ 32 \\ \hline 64 \\ 96 \\ \hline 1024=b^2 \\ 2 \\ \hline 2048=2b^2 \\ 576=y^2 \\ \hline 2624=2b^2+y^2 \end{array} \qquad \begin{array}{r} 24=y \\ 24 \\ \hline 96 \\ 48 \\ \hline =576 \end{array} \qquad \begin{array}{r} 107,51518=Q \\ 3 \\ \hline l=40)\ 322,54554=3Q \\ (8,06364=\frac{3Q}{l} \\ 8,00000=b-y \\ \hline -,06364=b-y-\frac{3Q}{l} \\ 288=8d \\ \hline 50912 \\ 50912 \\ 12728 \\ \hline \end{array}$$

$$-18,32832=8d\times b-y-\frac{3Q}{l}=18,32832$$

$$\begin{array}{r} 2605,67168 \\ 40 \\ \hline 104226,8672 \\ 8839971 62,0 \\ \hline 208453734 \\ 62536120 \\ 1042269 \\ 729588 \\ 93804 \\ 9380 \\ 313 \\ 83 \\ 8 \\ \hline \end{array}$$

$2605,67168 = 2b^2+y^2+8d\times b-y-\frac{3Q}{l}$

$8839971 62,0$ $(=a)$ *inverted.*

$$27286,5299=\overline{2b^2+y^2+8d\times b-b-\frac{3Q}{l}}\times\frac{al}{3}$$

And

And for the equilateral *Hyperbolic Spindle* this

OPERATION.

$105,80839 = Q$

3

$l = 40)\,317,42517 = 3Q$

$(7,93563 = \frac{3Q}{l}$

$8,00000 = b - b$

$0,06437 = b - b - \frac{3Q}{l}$

$512 = 8d$

12874

6437

32185

$32,95744 = 8d \times \overline{b - b - \frac{3Q}{l}}$

$2624\ \ldots = 2b^2 + b^2$

2591,04256

$40 = l$

$103641,70240 = \overline{2b^2 + b^2 - 8d \times \overline{b - b - \frac{3Q}{l}}} \times l$

883997162,0 = 261799, &c. inverted.

2072834048

621850214

10364170

7254919

932775

93277

3109

829

83

$27133,33424 = \overline{2b^2 + b^2 - 8d \times \overline{b - b - \frac{3Q}{l}}} \times \frac{al}{3}$

I must observe the Value of Q in the Rule for measuring the Hyperbolic Spindle, might have been computed from Mr. *Coats's* Theorem for that purpose, *Page* 110; but the Series in this Case does it with less trouble.

Or, By the two Theorems for measuring these Solids by Approximation, we have for the

$$\left.\begin{array}{ll}\text{1st.} & \overline{2624 - \frac{14.64.1152}{56320}} \times ,2618 \times 40 = 27286,53 \\ \text{2d.} & \overline{2624 - \frac{14.64.2048}{55680}} \times ,2618 \times 40 = 27133,35\end{array}\right.$$

Nearly the same as before.

Scholium I.

We shall now collect together the several Theorems found by *Proposit.* 2, 5, and 6; which include all the Varieties of Casks, hitherto mentioned by those who have wrote on Gauging, and set down the Theorems, and their Contents for Mr. *Everards's* Cask, whose Bung-Diameter is $b=32$, Head $b=24$, and Length $l=40$ Inches: The Names of which are, *1st*, The Middle Frustums of a *Spheroid*; *2d*, *Circular Spindle*; *3d*, *Hyperbolic Spindle*; *4th*, *Parabolic Spindle*; 5th, Two Frustums of a *Parabolic Conoid*; and *6th*, two Frustums of a *Cone*.

			Cub. Inch.	Ale Gall.
1	$\overline{2b^2+b^2}\times\frac{al}{3}$	=	27478,464	= 97.4414
2	$\overline{2b^2+b^2+8d\times\overline{b-b-\frac{3Q}{l}}}\times\frac{al}{3}$	=	27286,530	= 96.7607
3	$\overline{2b^2+b^2-8d\times\overline{b-b-\frac{3Q}{l}}}\times\frac{al}{3}$	=	27133.334	= 96.2175
4	$\overline{2b^2+b^2-\frac{2}{5}\times\overline{b-b}^2}\times\frac{al}{3}$	=	27210,381	= 96.4907
5	$\overline{b^2+b^2}\times\frac{al}{2}$	=	25132,741	= 89.1232
6	$\overline{b^2+b\times b\times\overline{b\times b}}\times\frac{al}{3}$	=	24797,638	= 87.9349

The Meaſures here differ a ſmall matter from what Mr. *Everard* gave in his *Appendix*, which ariſes from his inductionary Method of computing their Contents, but the true Theorems for Meaſuring the circular, elliptic and hyperbolic *Spindles* were never, that I know of, publiſh'd before, if we except the laſt, for which a Theorem is given by a Gentleman (that writes himſelf *Merones*) in the *Mathematical Diary* for 1740.

I muſt obſerve the Hyperbola in the above Example was ſuppoſed to be Equilateral, becauſe otherwiſe the *Data* was not ſufficient, but the Theorem

orem itself is universal, for all Hyperbolic Spindles whatever.

SCHOLIUM II.

To what has been already said for measuring Casks, we may add the Method for finding what is call'd the *mean Diameter*; for tho' Rules for that purpose are to be met with in most Books on Gauging, yet I cannot find any that have shewn whence they were derived, nor is this Defect all that makes it necessary here to examine into this Matter, but chiefly to prevent the Errors that must unavoidably arise from using those fixt Numbers call'd *Multipliers*: for there neither are, nor is it possible there should be such, that will reduce all Casks of the same Name and Length, to Cylinders, but they ought to vary as the Proportion betwixt the Bung-Diameter, and the Difference of this from the Head varies. 'Tis true indeed that Proportion may be supposed pretty near the same in most Casks to be met with in Practice, which is all that many are solicitous about; this I allow, however, since they are not absolutely so, 'tis necessary, nay I am persuaded every Officer of the least Ambition, to have a *rational* Knowledge of the Art he professes, must be desirous to know when the fixt Multipliers may be depended on, and when not, and also whence they were obtained.

That ready *Analyst*, Mr. *Tho. Simpson*, in his Treatise of Fluxions, has given the *Limits* betwixt which are the Numbers we are speaking of, and from what he has done, with little Trouble the very Multipliers themselves may be found: However, as his Book may not be in the Hands of every one concern'd in Gauging, and also some of them may want the necessary Knowledge to

make ſuch Inferences, I have not only given the *very Rules themſelves* for finding the Multipliers, but alſo the *Limits* from a different Conſideration, which I ſuppoſe will be underſtood with leſs Trouble, as I have no occaſion for the Term *Infinite*, which that Gentleman is oblig'd to introduce in determining one of them.

In order to which, let b be the Bung-Diameter of any Caſk, in the Shape of the Fruſtum of a *Spheroid*, *Parabolic Spindle*, *Parabolic Conoid*, or *Cone*, b the Head Diameter, and $d = b - b$: Alſo ſuppoſe m ſuch a Number or Quantity, that multiplying $b - b$, or d, and the Product *taken from* b, the Bung, the Remainder may be the Diameter of a Cylinder, equal in Length to the Caſk propos'd and of equal Magnitude; and put m' to denote the Multiplier when the Product is to be *added* to b, the Head Diameter, in order to have the *Mean* ſought. Then 'tis evident $m' = 1 - m$, for ſince $b - m \times \overline{b - b} = b + m' \times \overline{b - b}$, we have $b - b = \overline{m + m'} \times \overline{b - b}$, or $m' = 1 - m$.

But to find m (by the preceding Scholium and *Prop.* 1. *Chap.* 4.) we have theſe Equations.

$$\text{viz. For two Fruſtrums of a} \left\{ \begin{array}{l} \textit{Spheroid} \\[1ex] \textit{Parab. Spin.} \\[1ex] \textit{Parab. Con.} \\[1ex] \textit{Cone} \end{array} \right| \begin{array}{l} \dfrac{2b^2 + b^2}{3} \\[1ex] \dfrac{2b^2 + b^2 - \frac{2}{5}d^2}{3} \\[1ex] \dfrac{b^2 + b^2}{2} \\[1ex] \dfrac{b^2 + b^2 + bb}{3} \end{array} \left. \vphantom{\begin{array}{l} a \\[1ex] a \\[1ex] a \\[1ex] a \end{array}} \right\} = \overline{b - md}\,^2$$

In which for b put its value $b - d$, and for $\frac{b}{b - b}$ write r; then from the Equations above, we ſhall reſpectively have the following ones, *viz.*

$\frac{1}{r} =$

$$\frac{1}{r} = \left\{ \begin{array}{c} \frac{6m^2-2}{3m^2-1} \\ \frac{30m-10}{15m^2-3} \\ \frac{4m^2-2}{2m^2-1} \\ \frac{6m-3}{3m^2-1} \end{array} \right\} \text{and } m = \left\{ \begin{array}{l} r-\sqrt{r^2+\frac{1-2r}{3}} = \frac{1}{3}-\frac{1}{3}\times\frac{b-b}{2b+b} \\ r-\sqrt{r^2+\frac{3-10r}{15}} = \frac{1}{3}-\frac{4}{15}\times\frac{b-b}{5b+b} \\ r-\sqrt{r^2+\frac{1-2r}{2}} = \frac{1}{2}-\frac{1}{4}\times\frac{b-b}{b+b} \\ r-\sqrt{r^2+\frac{1-3r}{3}} = \frac{1}{2}-\frac{1}{12}\times\frac{b-b}{b+b} \end{array} \right\} \textit{proxime.}$$

And these Values of m taken from *Unity*, leave those of m' as hereunder,

$$\text{Two Frustums of a} \left\{ \begin{array}{l} \textit{Spheroid} \\ \textit{Parabol. Spindle} \\ \textit{Parab. Con.} \\ \textit{Cone} \end{array} \right\} m' = \left\{ \begin{array}{l} \frac{2}{3}+\frac{1}{3}\times\frac{b-b}{2b+b} \\ \frac{2}{3}+\frac{4}{15}\times\frac{b-b}{5b+b} \\ \frac{1}{2}+\frac{1}{4}\times\frac{b-b}{b+b} \\ \frac{1}{2}+\frac{1}{12}\times\frac{b-b}{b+b} \end{array} \right\} \textit{nearly.}$$

From hence it appears there can be no fixt Multipliers; but let us suppose in the most common Casks, the Bung-Diameter to be 32 Inches, and Head 26, then for the Frustum of a

Spheroid		$\frac{2}{3}+\frac{1}{3}\times\frac{6}{190}=,6888,$ &c.	
Parab. Spindle	$m'=$	$\frac{2}{3}+\frac{4}{15}\times\frac{6}{180}=,6753,$ &c.	*nearly.*
Parab. Con.		$\frac{1}{2}+\frac{1}{4}\times\frac{6}{58}=,5259,$ &c.	
Cone		$\frac{1}{2}+\frac{1}{12}\times\frac{6}{58}=,5087,$ &c.	

Which will serve indifferently, when the Bung-Diameter is betwixt *four* and *five* times the Difference betwixt it and the Head.

But those (,7 ; ,68 ; ,54 ; ,52) given by Mr. *Bamford*, can never answer, unless where the Bung-Diameter is betwixt *three* and *four* times the Difference betwixt that and the Head.

From what has been said, the *Limits* of all the Values of m, are easily deduced, by putting for b, its *extreme Values* in the second Series of Equations ; now they are b and o, for 'tis evident b must

must ever be greater than the greatest, and o less than the least, both which being substituted for b, we have

$$\frac{o}{b} = \left\{ \begin{array}{c} \frac{6m-2}{3m^2-1} \\ \frac{30m-10}{15m^2-3} \\ \frac{4m-3}{2m^2-1} \\ \frac{6m-3}{3m^2-2} \end{array} \right\} \text{and} \left\{ \begin{array}{c} \frac{6m-2}{3m^2-1} \\ \frac{30m-10}{15m^2-3} \\ \frac{4m-3}{2m^2-1} \\ \frac{6m-3}{3m^2-2} \end{array} \right\} = 1$$

And finding m in these Equations, 'tis manifest all Values thereof for the respective Varieties *Pag.* 164, must be

$$\text{Betwixt} \left\{ \begin{array}{l} \frac{1}{3} = ,3333, \text{ \&c.} \\ \frac{1}{3} = ,3333, \text{ \&c.} \\ \frac{1}{2} = ,5 \\ \frac{1}{2} = ,5 \end{array} \right\} \text{and} \left\{ \begin{array}{l} 1-\sqrt{\frac{2}{3}} = ,1835 \\ 1-\sqrt{\frac{8}{15}} = ,2697 \\ 1-\sqrt{\frac{1}{2}} = ,2927 \\ 1-\sqrt{\frac{1}{3}} = ,5774 \end{array} \right\} \text{\&c.}$$

And

And these Limits taken from *Unity*, leave those of m', which therefore are

$$\text{Betwixt}\left\{\begin{array}{l} \frac{2}{3}=,6666, \&c. \\ \frac{2}{3}=,6666, \&c. \\ \frac{1}{2}=,5 \\ \frac{1}{2}=,5 \end{array}\right\} \text{and} \left\{\begin{array}{l} \sqrt{\frac{2}{3}}=,8165 \\ \sqrt{\frac{8}{15}}=,7303 \\ \sqrt{\frac{1}{2}}=,7071 \\ \sqrt{\frac{1}{3}}=,5774 \end{array}\right\} \&c.$$

As they were first of all determin'd by Mr. *Simpson*, in his Treatise above mention'd.

From the foregoing Theorems the Reader will be able, with very little trouble, to find a *true* Multiplier for reducing any of the Forms above into a Cylinder of equal Length; but as I have shewn how they may be measur'd by the Lines C, D, at *one setting of the Sliding-Rule*, there can never be occasion for such Reductions: nay, I can always gauge any of those Casks sooner by those Rules, than by first reducing them to Cylinders. So that these Rules are really of no Service in gauging a *full Cask*, but will be of particular Use when we come to *Ullaging*, which is consider'd in the next Part.

CHAP. V.

Of measuring the Hoofs *of Pyramidic Cones,*

DEFINITION I.

IF a Point V (*Fig.* 51.) be situate in a fixt Position above the Plane ACEDB (of any Figure whatever) and from V as a Pole an Indefinite right Line be carried round the same, it will describe the *Likeness* of a Solid, which we call a *Pyramidic Cone*; of which ACEDB is the *Base*, and V its *Vertex*.

Defin. II. If a pyramidic Cone be cut by a Plane parallel to its Base, the Part contain'd betwixt the said Plane and Base, is call'd its *Frustum*.

Defin. III. If a Plane be drawn thro' the Vertex of the Cone perpendicular to its Base, its Section with the Frustum is call'd the *vertical Plane* thereof; and then its Sections with the *End* and *Base*, their *Diameters* respectively.

Defin. IV. If a Plane be drawn through either Extremity of the End's Diameter, at right Angles to the vertical Plane of the Frustum, and so as to cut the Cone's Base, the Part of the Frustum contain'd betwixt the cutting Plane and Part of the Base, is call'd a *Hoof* thereof; and the remaining Part of the Frustum, the *complemental Hoof*; also that Extremity of the End's Diamemeter through which the Cutting-Plane passeth, may be call'd the *Vertex of the Hoof*.

Defin. V. The *Containing* Part of the Cone's Base, is call'd the *Base of the Hoof*; the Section of the Cutting-Plane and Frustum, its *Side*; and the Section of the said Side and vertical Plane, the *Axis* of the Side.

LEMMA

LEMMA I.

If a Pyramidic Cone be cut by a Plane parallel to its Base, the Area of the Section will be to that of the Cone's Base, as the Square of the Perpendicular from the Vertex to the Plane of the Section, is to the Square of the Perpendicular's Height of the Cone.

For let VEQ be the vertical Plane of the Cone, which cuts the Base in EQ, and the Section *aeq* in *eq*, then EQ and *eq* are the Diameters of the Base and Section *aeq*, (*Definit.* 3.) on which from V let fall the Perpendicular VP, meeting the first in P, and the other in *p*, then V*p* is the vertical Distance of the Section. Also in EQ assume any Point F, and suppose the Distance EF divided into an infinite Number of equal Parts in the Points G, H, K, &c. thro' every one of which and the Vertex V, let Planes be supposed drawn perpendicular to VEQ, cutting the Base in the Lines CD, MM, NN, OO, &c. and the Section *aeq* in the same manner in *cd*, *mm*, *nn*, *oo*, &c. thereby distinguishing each into an infinite Number of Elementa; let those of the Base from F towards E be denoted by E', E'', E''', &c. and the corresponding ones of the Section *aeq* e', e'', e''', &c. then (*per Fig.*) $E' = CD \times FG$ and $e' = cd \times fg$, but $FG : fg :: FE : fe :: FP : fp$, and $FP : fp :: VP : Vp$; hence $E' : e' :: CD \times VP : cd \times Vp$. Again, (*from the Fig.*) $CD : cd :: VP : Vp$, therefore $E' : e' :: \overline{VP}^2 : \overline{Vp}^2$; and for the same Reason $E'' : e'' :: \overline{VP}^2 : \overline{Vp}^2$; also $E''' : e''' :: \overline{VP}^2 : \overline{Vp}^2$; whence (by the 12 *Eucl.* 5.) $E' + E'' + E'''$, &c. $e' + e'' + E'''$, &c. $:: \overline{VP}^2 : \overline{Vp}^2$: But $E' + E'' + E'''$, &c. $=$ CED, and $e' + e'' + e'''$, &c. $=$ *ced*, therefore CED : *ced* $:: \overline{VP}^2 : \overline{VP}^2$. *Q. E. D.*

Cor.

Cor. Whence it is evident all similar Planes are to another, as the Squares of their homologous Lines.

LEMMA II.

The Measure of every Pyramidic Cone is equal to the Product of the Measure of its Base, by one third Part of its perpendicular Height.

For let the Measure of the Cone's Base be b, its perpendicular Height VP$=h$, and x the Measure of Vp (see *Figure* 51.) then by the preceding Lemma $h^2 : b :: x^2 : \frac{bx^2}{h^2}$, the Measure of the Section $acedb$. Further, let us imagine Vp divided into an infinite Number of equal Parts, every one of which is measured by q, and through each of them let Planes be supposed drawn parallel to the Base of the Cone, thereby dividing the Part Vaeb into an infinite Number of cylindrical Elementa, and let their successive Distances from V be denoted by x, x', x'', x''', &c. *viz.* $x-q=x'$, $x'-q=x''$, $x''-q=x'''$, &c. then from what was said above, the successive Measures of the Bases of these elementary Cylinders will be $\frac{bx^2}{h^2}$, $\frac{bx'^2}{h^2}$, $\frac{bx''^2}{h^2}$, $\frac{bx'''^2}{h^2}$, &c. and therefore the Measures of the Cylinders themselves are $\frac{bx^2}{h^2}\times q$, $\frac{bx'^2}{h^2}\times q$, $\frac{bx''^2}{h^2}\times q$, $\frac{bx'''^2}{h^2}\times q$, &c. whence the Measure of the Part Vabc, is equal to the Sum of an infinite Series of Squares, whose Roots, x, x', x'', x''', &c. are in Arithmetic Progression from o to x inclusive, multiplied by $\frac{b\times q}{h^2}$, but the Number of them is $\frac{x}{q}$, whence (by the Lemma

pag. 85.) V*abc* is equal to $\frac{b \times q}{b^2} \times x^2 \times \frac{x}{q} \times \frac{1}{3} = \frac{bx^3}{3b^2}$, which when x becomes b, gives VAEB $= b \times \frac{b}{3}$. *Q. E. D.*

Lemma III.

Let GHI (Fig. 52.) *be any Right-lined Triangle, and in* HG *a Point* E *taken, and the Line* IE *drawn; also through* E *let* EK *be drawn parallel to* IG, *and made equal to* EI, *and through the Point* K *a Line drawn parallel to* HI *meeting* GH *produc'd in* V: *then a Perpendicular let fall thence on* IE (*continued if needful*) *will be equal to a Perpendicular drawn from* H *on* IG, *that is* VF=HP.

For draw IQ perpendicular to HG, then from the similar Triangles HGP, IGQ, we have GH : GI : : HP : IQ, but from the similar Triangles HIG, VKE, GH : GI : : EV : EK = EI (*per Constr.*) hence HP : QI : : EV : EI. Again, from the Triangles EIQ, EVF, we have IQ : IE : : VF : VE; whence, and the last Proportion, VF : VE : : HP : VE, therefore VF=HP. *Q. E. D.*

Prop. I.

Let KEBA (Fig. 53.) *be the Frustum of a Pyramidic Cone, whose Pole is* V, *Base the Plane* AMBN, *End the Plane* KE, *Vertical Plane* VAB, *and* MNEB *a Hoof thereof: then* AB, KE *are the Diameters of the Base and End respectively, and* EL *the Axis of the Side; on which make* EI *equal to* EK, *and through* I *draw* IG, IH *parallel to* AB, AV *respectively; and from the Points* V, H, *let fall* VC, HP *Perpendiculars on* AB, IG; *then* (*calling the Base of the Hoof* MNB, S, *and its Side* EMN, S') *the Measure of the Hoof* MNEB *is expressed by this*

Theorem,

THEOREM, *viz.*

$$MNEB = \overline{BC \times S - GP \times S'} \times \frac{ER}{3BR}.$$

For through V and the Line MN, imagine a Plane drawn, whose Section with the Cone is the Triangle VMN, and from V let fall a Perpendicular on the Axis of the Side (produc'd if necessary) meeting the same in F.

Now 'tis manifest the Hoof MNEB is equal to the pyramidic Cone VMNB, made less by VMNE, but (by *Lemma* 2.) $VMNB = S \times \frac{VC}{3}$, and $VMNE = S' \times \frac{VF}{3}$. But $VC = \frac{ER \times BC}{BR}$ (for BR : ER :: BC : VC) also VF=HP by *Lemma* 3, and $HP = \frac{ER \times GP}{BR}$ (for BR : ER :: GP : HP) therefore for VC and VF in the above Values of VMNB and VMNE put their equals, and we have

$$MNEB = \overline{BC \times S - GP \times S'} \times \frac{ER}{3BR}. \quad \textit{Q. E. D.}$$

Cor. 1. But if the Cone be upright, or the Triangle VAB isosceles, then $BC = \frac{1}{2}AB$, and $GP = \frac{1}{2}IP$; whence if D be put to denote the Diameter of the Base, and d that of the End of the Frustum, *viz.* $D = AB$, $d = KE$, and $ER = b$, the Height of the Frustum, then $BC = \frac{D}{2}$, $GP = \frac{d \times BL}{2EL}$, and $BR = \frac{D - d}{2}$; whence, in this Case, the Measure of the Hoof MNEB is express'd by this

THEOREM, *viz.* $\overline{D \times S - \frac{d \times BL}{EL} \times S'} \times \frac{b}{3 \times \overline{D - d}}$.

Cor.

Cor. 2. Let $M\beta\beta N k\epsilon\epsilon k$ (*Fig.* 53.) be the Frustum of an upright square Pyramid, which is cut through the Sides $\epsilon\epsilon$, MN by a Plane $\epsilon\epsilon MN$; then if D be the Side of the Base, d that of the End, and b the perpendicular Height of the Frustum, the Measure of the

$$\left\{\begin{array}{ll} \textit{Hoof} & \epsilon\epsilon MN\beta\beta = \overline{2D+d} \times \frac{bD}{6} \\ \textit{Com. Hoof} & \epsilon\epsilon MN kk = \overline{2d+D} \times \frac{bd}{6} \end{array}\right\}$$

For since the (Pyramidic Cone, or) Pyramid is upright, (by *Coroll.* 1.) we have $\epsilon\epsilon MN\beta\beta =$ $\overline{D \times S - \frac{d \times BL}{EL} \times S'} \times \frac{b}{3.\overline{D-d}}$, but $S = D^2$, and $S' = \frac{D+d}{2} \times EL$, as is evident from the Figure, whence $\frac{d \times BL}{EL} \times S' = \frac{d \times D}{EL} \times \frac{D+d}{2} \times EL = dD \times \frac{D+d}{2}$, therefore $\overline{D \times S - \frac{d \times BL}{EL} \times S'} \times \frac{b}{3.\overline{D-d}}$ becomes $\overline{D^3 - \frac{1}{2}Dd \times \overline{D+d}} \times \frac{b}{3 \times \overline{D-d}} = \frac{2D^3 - D^2 d - Dd^2}{D-d} \times \frac{b}{6}$, but $\frac{2D^3 - D^2 d - Dd^2}{D-d} = 2D^2 + Dd$, hence $\epsilon\epsilon MN\beta\beta = \overline{2D+d} \times D \times \frac{b}{6}$: and this taken from $\overline{D^2 + Dd + d^2} \times \frac{b}{3}$, the Area of the whole Frustum (as easily follows from *Lemma* 2.) leaves $\epsilon\epsilon MN kk = \overline{2d+D} \times d \times \frac{b}{6}$. *Q. E. O.*

And from hence these Rules, to measure the Hoofs of the Frustum of a square Pyramid.

RULE

RULE I.

To twice the Side of the Baſe add that of the End, multiply the Sum by the Side of the Baſe, and that by one ſixth of the Height of the Hoof, the Product will be its Content.

RULE II.

To twice the Side of the End add that of the Baſe, the Sum multiplied by the Side of the End, and that Product by one ſixth of the Fruſtum's Height, gives the Content of the *Complemental Hoof*

By the SLIDING-RULE.

RULE III.

Set 3,464 on D to the Fruſtum's Height on C, and find the Side of the Baſe, that of the End, and their Difference on D, noting the three Numbers oppoſite them on C. Then to the ſecond add five times the firſt, and from that Sum take the third, the Remainder is the Meaſure of the *Hoof*.

But if to the firſt five times the ſecond be added, and the third taken from that Sum, the Remainder is the Meaſure of the *Complemental Hoof*.

Example. Let there be the Fruſtum of a ſquare Pyramid, the Side of whoſe Baſe is 25, that of its End 16, and the Height of the Fruſtum 12 Inches, which is cut by a Plane through oppoſite Sides of its End and Baſe, to find the Meaſures of both the Hoofs.

OPERATION.

OPERATION.

$$\left\{\begin{array}{l} D=25 \\ d=16 \\ b=12 \end{array}\right\}$$

$50=2D$	$32=2d$
$16=d$	$25=D$
$66=2D+d$	$57=2d+D$
25	$16=D$
330	342
132	57
$1650=\overline{2D+d}\times D$	$912=\overline{2d+D}\times d$
$2=\frac{1}{6}b$	$2=\frac{1}{6}b$
$3300=\overline{2D+d}\times D\times\frac{1}{6}b$	$1824=\overline{2d+D}\times d\times\frac{1}{6}b$
The Meas. of the Hoof.	*The Meas. of the com. Hoof.*

Or, 3,464 on D set to 12 on C, then against 25, 16, and 9 on D is 625, 256 and 81 on C respectively; therefore by the Rule,

$$\left\{\begin{array}{l} 3125\ (=625\times 5)+256-81=3300 \\ 1280\ (=256\times 5)+625-81=1824 \end{array}\right\}\textit{as before.}$$

Cor. 3. If ABKE (*Fig.* 54.) be the Frustum of an Upright Cone with a circular Base, whose vertical Plane is ABKE, which is cut by a Plane thro' the Points E, A, and D be the Diameter of the Base, d that of the End, and b its Height, then the Measure of

$$\text{The}\left\{\begin{array}{l} \textit{Hoof}\ \text{ABE} \\ \textit{Com. Hoof.}\ \text{AEK} \end{array}\right. = \begin{array}{l} \overline{D^2-d\sqrt{Dd}}\times\dfrac{D\times b\times a}{3\times\overline{D-d}}, \\ \overline{D\sqrt{Da}-d^2}\times\dfrac{d\times b\times a}{3\times\overline{D-d}}. \end{array}$$

For

For here the Points M, N and L, are united in A, and the Space S′ (by *Defin.* 3. *Chap.* 2.) is an Ellipse, whose Transverse is EA or EL, and Conjugate a mean Proportional betwixt D, d; hence $S' = \overline{EL} \times \sqrt{Dd} \times a$, (*Cor.* 3. *Prop.* 4. *Chap.* 3.) and $S = D^2 \times a$. Put these Values for S and S′ in the *Theorem Cor.* 1. then the Measure of the Hoof is $\overline{D \times D^2 \times a - \frac{d \times D \times EL \times \sqrt{Dd} \times a}{EL}} \times \frac{b}{3 \times \overline{D-d}} =$ $\overline{D^2 - d\sqrt{Dd}} \times \frac{D \times b \times a}{3 \times \overline{D-d}}$; which taken from $\overline{D^3 - d^3} \times \frac{ab}{3 \times \overline{D-d}}$, the Measure of the whole Frustum, (see *Cor.* 3. *Prop.* 2. *Chap.* 3.) leaves $\overline{D\sqrt{Dd} - d^2} \times \frac{d \times b \times a}{3 \times \overline{D-d}}$, for the Measure of the *Complemental Hoof.* *Q. E. O.*

And from hence we have these two *Rules for measuring the elliptic Hoofs of an upright Cone.*

RULE I.

From the Square of the Diameter of the Base, substract the Product of the Diameter of the End by a mean Proportional betwixt the two Diameters; multiply the Remainder by the Diameter of the Base, and that Product by the Height of the Frustum, and this by ,261799388 (or in Practice by ,2618) then if the last Product be divided by the Difference of the Diameters, the Quotient will be the Measure of the *Hoof* proposed.

RULE II.

From the Product of the Diameter of the Base by a mean Proportional betwixt that and the Diameter of the End, substract the Square

of the Diameter of the End, multiply the Remainder by the Side of the said Square, and that Product by the Height of the Frustum, and this by ,261799388 (or ,2618) then if the last Result is divided by the Difference of the two Diameters, the Quotient will be the Measure of the *Complemental Hoof.*

Examp. Let the Diameter of the Base be 60, that of the End 38,4, and the Height 36 Inches,

Operation *for* ABE (*Fig.* 55.)

$D = \ldots 60$

$d = \ldots 38{,}4$

$D \times d = 2304{,}0 \quad (48 = \sqrt{Dd}\,) = D - d = 21{,}6)\,3794688 \,|\, 275680{,}00$

```
        16               38,4=d                216     4997162,0
      ----              -------               -----   ----------
   8) 704               3072                  1634      3513600
      704               1536                  1512      1054080
      ----              ------               -----       17568
        0               1843,2=d√Dd            1226       12298
      ----              3600  =D²             1080        1581
                        ------                -----        158
                        1756,8=D²−d√Dd         1468          7
                          60=D                 1296     ---------
                      -----------             -----     45992,92=
                      105408,0=D²−d√Dd×D       1728     Hoof ABE.
                          36  =h               1728
                      -----------             -----
                        632448                 0000
                       316224                 -----
                      -----------
                       3794688  =D²−d√Dd×D×h
```

OPERATION for AKE (*Fig.* 55.)

$38,4 = d$
384
1536
3072
1152
$1474,56 =$

48 $= \sqrt{Dd}$
60 $= D$
2880 $= D\sqrt{Dd}$
1474,56 $= d^2$
1405,4 $= D\sqrt{Dd} - d^2$
38,4 $= d$
56217 6
1124352
421632
53968,896 $= d \times \overline{D\sqrt{Dd} - d^2}$
36 $= b$
323813376
161906688

$D - d = 21,6$) 1942880,256 (89948,16

2148	4997162
2048	17989632
1040	5396890
1762	89948
345	62964
1296	8095
0000	810
	36
	23548,375 = *Comp. Hoof* AKE

We might alſo ſhew how to meaſure theſe Hoofs by the *Sliding-Rule*. But the Operation would be too tedious for Practice, ſo I proceed to the *Parabolic Hoof*.

Cor. 4. The ſame being ſuppoſed as in the laſt Corollary; only here let EL (*Fig*. 56.) the Axis of the Hoof's Side, be parallel to KA; then MEN the Side it ſelf (by *Defin*. 6. *Chap*. 2.) is a Parabola, and therefore the Meaſure of the

$$\left.\begin{array}{l}\textit{Hoof}\ \mathrm{EBL}\\ \textit{Com.}\ \textit{Hoof}\end{array}\right\} = \left\{\begin{array}{l}\overline{\frac{\mathrm{D}}{\mathrm{D}-d}\times\mathrm{S}-\frac{4}{3}d\sqrt{d\times\overline{\mathrm{D}-d}}}\times\frac{b}{3}\\ \overline{\mathrm{D}^2+d\times\overline{\mathrm{D}+d}\times a-\frac{\mathrm{D}\times\mathrm{S}}{\mathrm{D}-d}+\frac{4}{3}d\sqrt{d\times\overline{\mathrm{D}-d}}}\times\frac{b}{3}\end{array}\right.$$

For

For S′ (by *Prop*. 4. *Chap*. 3.) $= \frac{2EL \times MN}{3} = \frac{4EL}{3}\sqrt{d \times \overline{D-d}}$, therefore $\frac{d \times BL}{EL} \times S' = d \times BL \times \frac{4}{3}\sqrt{d \times \overline{D-d}}$, or $\frac{4}{3} \times d \times \overline{D-d}\sqrt{d \times \overline{D-d}}$; let this be put in its Room in the Theorem *Cor*. 1. and it becomes $\overline{D \times S - \frac{4}{3} \times d \times \overline{D-d}\sqrt{d \times \overline{D-d}}} \times \frac{b}{3 \times \overline{D-d}} = \frac{D}{D-d} \times S - \frac{4d}{3}\sqrt{d \times \overline{D-d}} \times \frac{b}{3}$, and this taken from $\overline{D^2 + d \times \overline{D+d}} \times \frac{ab}{3}$ the whole Fruſtum, leaves the Meaſure of the *Complemental Hoof* as above. *Q. E. O.*

Exampl. Let the Dimenſions of a Conic Fruſtum be as above, to find the Meaſure of the *Parabolic Hoof.*

Firſt we muſt find S the Segment of a Circle, whoſe Diameter is D=60, and verſed Sine D—*d* =21,6: Whence by the Rule, *Page* 105, I annex two Cyphers to 21,6, that ſo it may conſiſt of three Decimals as the Rule requires, and it becomes 21,600, which divided by 60, gives ,360, againſt which in *the Table of Segments* is ,2545505; this multiplied by 3600, the Square of 60, gives 916,3819=S, alſo $\frac{D}{D-d} = \frac{100}{36}$, therefore $\frac{D}{D-d} \times S = \frac{100S}{36}$, and from hence the following

OPERATION for ELB, Fig. 56.

$d = 38.4$
$D - d = 21.6$

```
    2304
   384
  768
```

$\sqrt{d \times \overline{D-d}} = \sqrt{829,44} = 28,8$

```
   4
4) 429
   384
56) 4544
    4544
    0000
```

$51,2 = \frac{4d}{3}$

```
  576
 288
1440
```

$1474,56 = \frac{4d}{3}\sqrt{D \times \overline{D-d}}$

```
36)91638,19 | 2545,5053
   196      | 
    163     | 
     198    | 
      181   | 
       190  | 
```

$2545,5053 = \frac{100S}{36}$

$1474,56$

$1070,9453$

$12 = \frac{b}{3}$

$12851,3436$

Hence

Hence the Measure of the *Parabolic Hoof* is 12851,3436; and this taken from $\overline{D^2+d\times D+d} \times \frac{ab}{3} = 69541,2893$, the Measure of the whole Hoof, leaves 56689,9457 for the *Complemental Hoof* ELB.

Cor. 5. The same things being still supposed, only here let MEN (*Fig.* 57.) be an Hyperbola, then the *Measure of the Hyperbolic Hoof* MNEB, *is expressed by this*

THEOREM, *viz.*

$$MNEB = \overline{D \times S - \frac{d \times LB}{LE} \times S'} \times \frac{b}{3 \times \overline{D-d}}.$$

Where LE is the Diameter of the Side, and LB that of the Base, which must be given before these Hoofs can be measured: But if MEN, the Cutting Plane, should be Perpendicular to the Base of the Cone, then the *Measure of the Hoof* MNEB *is expressed by this*

THEOREM, *viz.*

$$MNEB = \frac{Db}{3 \times \overline{D-d}} \times S - \frac{d}{6} \times S'.$$

And the transverse Axis of the Hyperbola, S′ or MEN, is $\frac{2db}{D-d}$, Conjugate $\frac{d}{2}$.

The first Theorem is evident from *Corol.* 1, and the second follows from that; for when MNE is perpendicular to the Base of the Cone, then $LB = \frac{D-d}{2}$, and $LE = b$, which being put for LB and LE, will give the Theorem above.

But as $\frac{D-d}{2}$ is commonly small in respect of D, we may for those Cases give an *Approximation* of

Theorem 2, which is very near the Truth; for D, *d*, and *b* ſtanding for the ſame Sines as before,

$$MNEB = \frac{D \times \overline{47D + 23d}}{8D + 6d} - \frac{d \times \overline{47d + 23D}}{8d + 6D} \times \frac{b}{45} \sqrt{\overline{D^2 - d^2}} \text{ proximè.}$$

For an Example to theſe two Theorems, let $D = 32$, $d = 24$ and $b = 20$, therefore $BL = \frac{D - d}{2} = 4$.

In the firſt place we muſt find the Values of S and S′, that is the Area of the Segment of a Circle whoſe Diameter is 32, and verſed Sine 4; alſo the Area of an Hyperbola, whoſe tranſverſe Diameter is $\frac{2db}{D - d} = 120$, Conjugate 12, and Abſciſſa 20; for the firſt I annex three Cyphers to 4, and devide it by 32, the Quotient 125 I look for in the Table of Segments under VS, againſt which under *Seg.* is ,0566640; this multiplied by $\overline{32}|^2 = 1024$ gives $58.0239 = S$. And to find S′, (by *Prop.* 6. *Chap.* 3.) we muſt put for *x*, *y*, and V their Values, which here are 20, $\sqrt{4 \times 28}$, and 140; therefore calling $\frac{4yx}{3} = \frac{80}{3}\sqrt{112} = 282{,}21347 = A$; then $S' = A - \frac{A}{5 \cdot 7} - \frac{B}{7 \cdot 7} - \frac{3C}{9 \cdot 7} - \frac{5D}{11 \cdot 7}$, &c. as follows.

A =	282.21347
B =	8,06324
C =	16456
D =	784
E =	51
F =	5
Sum =	8,23620
S′ =	273,97727

There

Therefore for S and S′, put these Values in the second Theorem, then $\frac{Db}{3\times\overline{D-d}}\times S$, becomes $\frac{32\times 20}{3\times 8}$ $\times$ 58,0239 = ... 1547,3040

and $\frac{d}{6}\times S' = 4\times 273{,}97727 =$ 1095,9091

Their Difference is the Measure of EMNB = ... } 451,3949

But if we use the Approximation, then we shall have

$$\overline{32\times\frac{47\times 32+23\times 24}{8\times 32+6\times 24}-24\times\frac{47\times 24+23\times 32}{8\times 24+6\times 32}}\times\tfrac{20}{45}\sqrt{448}$$

that is EMNB $= \overline{164{,}48-116{,}5}\times\frac{4}{9}\sqrt{448} =$ 451,35, nearly the same as before.

So that from what has been said in this Proposition and its Corollaries, the Reader will be able to measure the Hoof of any Cone, let the Figures of the Base and Side be what they will; from whence is deduced the Method of Gauging an inclin'd Conical Tun. Also from the last Corollary any Cask in the Shape of two Frustums of a Cone may be ullaged; but in what follows, I shall lay down a Proposition for measuring Planes or Solids by Approximation, a thing of the greatest Importance to this Part of Science, of any that was ever brought for that Purpose, since it may be said to contain the *whole Art of Gauging*, and that of *Coppers*, *Stills*, *Tuns*, as well as all kinds of Casks, whether full or part empty, either standing or lying.

CHAP. VI.

Of measuring Curve-Lin'd Planes, and Solids, by Approximation.

PROP.

IF MQ$=y'$, NR$=y''$, PS$=y'''$ (Fig. 58.) *represent three equidistant perpendicular Ordinates to the Axis of a Curve* MNP, *whose Equation is* y$=$ a$+$bx$+$cx^2, *where* x *stands for any Abscis* QT, *and* y *its Ordinate* TO; *then, calling* QS *the Distance of the extream Ordinates* l, *the Measure of the Space* QMPS *shall be express'd by this*

THEOREM, *viz.* QMPS $=\overline{y'+y'''+4y''}\times\frac{l}{6}$.

For by the preceding Principles, the Quadrature of the Curve, whose Abscissa is x, and Ordinate $a+bx+cx^2$, will be found $\overline{a+\frac{bx}{2}+\frac{cx^2}{3}}\times x$; and this when x becomes l, is $\overline{a+\frac{bl}{2}+\frac{cl^2}{3}}\times l$ $=\overline{6a+3bl+2cl^2}\times\frac{l}{6}$, $=$ QMPS.

But from the Equation of the Curve we have these three Equations,
$$\left\{\begin{aligned} y'&=a\\ y''&=a+\frac{bl}{2}+\frac{cl^2}{4}\\ y'''&=a+bl+cl^2\end{aligned}\right.$$

And by taking the Difference of those above, we have these,
$$\left\{\begin{aligned} y''-a&=\frac{bl}{2}+\frac{cl^2}{4}\\ y'''-y''&=\frac{bl}{2}+\frac{3cl^2}{4}\end{aligned}\right.$$

And the Difference of the last gives
$$\left\{\, y'''-2y''+a=\frac{cl^2}{2}\right.$$

Whence

Whence $2cl^2 = 4 \times \overline{y''' - 2y'' + a}$; and by the 3d Equation above, $bl = y''' - a - cl^2$, therefore $3bl = 3y''' - 3a - 6 \times \overline{y''' - 2y'' + a} = -3y''' + 12y'' - 9a$; put these for $2cl^2$ and $3bl$, in the above Expression of the Area, and then we have

$$QMPS = \overline{6a + 3bl + 2cl^2} \times \frac{l}{6} = \overline{y' + 4y'' + y'''} \times \frac{l}{6}.$$

Q. E. D.

Cor. 1. The same Method of Demonstration extends to any Number of equidistant Ordinates; so if A denotes the Sum of the extream Ordinates, B the Sum of those next to them, C the Sum of the two next following the last, and so on; then we shall have the following *Tables of Areas*, for the several Numbers of Ordinates prefixt against them, *viz.* for

2	$A \times \frac{l}{2}$
3	$\overline{A + 4B} \times \frac{l}{6}$
4	$\overline{A + 3B} \times \frac{l}{8}$
5	$\overline{7A + 32B + 12C} \times \frac{l}{90}$
6	$\overline{19A + 75B + 50C} \times \frac{l}{288}$
7	$\overline{41A + 216B + 27C + 272D} \times \frac{l}{840}$
8	$\overline{751A + 3577B + 1323C + 2989D} \times \frac{l}{17280}$
&c.	&c.

This Method was invented by Sir *Isaac Newton*, and published by Mr. *Jones* in 1711; and since prosecuted by Mr. *Coats*, Mr. *De Moivre*, and by Mr. *Stirling*, in a whole Treatise entirely built thereon,

thereon, where such as desire a further Insight into this matter, may find it sufficiently explain'd, and applied to some of the most intricate *Parts* of Mathematicks.

But as I am going from thence to offer a new Method of Gauging, by which all *Guessing* will be laid aside, I judg'd it not amiss to shew its Use in some Instances of a different Nature.

And *First*, 'Tis known the *Hyperbolic Logarithm of any Number denoted by* $1+x$, *is as the Area of a Curve whose Abscissa is* x *and Ordinate* $\frac{1}{1+x}$; therefore to find this Area, or the Log. of $1+x$, let x be supposed equal to 1, and divided into six equal Parts, then we shall have the seven following equidistant Ordinates $\frac{1}{1+0}$, $\frac{1}{1+\frac{1}{6}}$, $\frac{1}{1+\frac{2}{6}}$, $\frac{1}{1+\frac{3}{6}}$, $\frac{1}{1+\frac{4}{6}}$, $\frac{1}{1+\frac{5}{6}}$, $\frac{1}{1+\frac{6}{6}}$; that is, $\frac{6}{7}$, $\frac{6}{8}$, $\frac{6}{9}$, $\frac{6}{10}$, $\frac{6}{11}$, $\frac{6}{12}$, and therefore according to the Rule, against the Ordinates in the Table of Areas, we have this Process;

A=

$A = 1 + \frac{6}{12} = \frac{18}{12} = 1{,}5$
$B = \frac{6}{7} + \frac{6}{11} = \frac{108}{77} = 1{,}4025974$
$C = \frac{6}{8} + \frac{6}{10} = \frac{27}{20} = 1{,}35$
$D = \frac{6}{9} + 0 = \frac{2}{3}, 666666$

and

$41A = 61{,}5$
$216B = 302{,}9610384$
$27C = 36{,}45$
$272D = 181{,}3333333$

Sum $582{,}2443717$

And the Sum 582,2443717, divided by 840, equal to $\frac{l}{840}$ (for $l = x = 1$) gives ,6931479, which is the *Hyperbolic Log.* of $2 = 1 + x$, and would have required at least 1000000 Terms of the common Series $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$, *&c.*

For a *Second Example*; Let it be required to square the Curve, whose Equation is $\frac{1}{1+t^2}$, this (by what was shewn *Prop.* 2. *Chap.* 1.) is equivalent to the circular Arch, whose Radius is 1, and Tangent t; let the Arch be 45 Degr. then $t = 1$,

$t=1$, which suppose divided into six equal Parts, and we have these seven equidistant Ordinates,

$$\frac{1}{1+0},\ \frac{1}{1+\frac{1}{36}},\ \frac{1}{1+\frac{4}{36}},\ \frac{1}{1+\frac{9}{36}},\ \frac{1}{1+\frac{16}{36}},\ \frac{1}{1+\frac{25}{36}},\ \frac{1}{1+\frac{36}{36}}$$

Or, $1. \frac{36}{37},\ \frac{36}{40},\ \frac{36}{45},\ \frac{36}{52},\ \frac{36}{61},\ \frac{36}{72}.$

And hence by the Rule against the Ordinates in the Table, this Process,

$$A = 1 + \tfrac{36}{72} = 1,5$$
$$B = \tfrac{36}{37} + \tfrac{36}{61} = 1,563137$$
$$C = \tfrac{36}{40} + \tfrac{36}{52} = 1,192308$$
$$D = 0 + \tfrac{36}{45} = 0,8$$

and

$$41A = 61,5$$
$$216B = 337,637592$$
$$27C = 42,992316$$
$$272D = 217,6$$

Sum, 659,729908

And this Sum divided by 840, gives ,785393 for the Length of the circular Arch of 45 Deg. erring but 5 in the last place,

These two Examples will be sufficient to give the Reader a Proof of the great Use and Accuracy of this Method in Quadratures; I shall now proceed

ceed to ſhew its Application in meaſuring of Solids.

Cor. 2. For if y', y'', y''' in the Propoſition, inſtead of the Ordinates MQ, NR, PS, ſhould denote the Circles generated by them, when the whole Curve MNP turns on QS as an Axis, then the Expreſſion itſelf would be the Meaſure of the Solid, generated from that Motion, which therefore is expreſs'd by $\overline{y'+4y''+y'''} \times \frac{l}{6}$, or putting d', d'', d''' for the Diameters of thoſe Circles, the Meaſure of the Solid will be $\overline{d'^2+4d''^2+d'''^2} \times \frac{al}{6}$.

And therefore when this Method is applied to meaſuring Solids, the Quantities A, B, C, D, &c. *Page* 187. denote Spaces, but Lines when uſed in meaſuring Spaces.

Cor. 3. So if S$'$ denote one End of a Solid, S$'''$ the other End, and S$''$ a Section thereof by a Plane in the Middle betwixt the two Ends, then the Meaſure of that Solid is nearly expreſs'd by $\overline{S'+4S''+S'''} \times \frac{l}{6}$, where l denotes the perpendicular Diſtance betwixt the Planes S$'$, S$'''$.

The ſecond *Corol.* is ſufficient for Meaſuring or Gauging all Solids, generated by a Curve revolving on ſome Line as an Axis; and the third for all others, or Ullaging a lying Caſk.

Cor. 4. For let the Bung and Head-Diameters of any Caſk be denoted by b, h, and the Length l, as before; alſo put m to denote a Diameter taken in the Middle betwixt the Bung and Head, then for meaſuring any Caſk whatever in cubic Inches, this is the

THEOREM, $\overline{b^2+h^2+4m^2} \times \frac{al}{6}$, *nearly.*

Or,

Or *by the Sliding-Rule*, set 2,764 on D to the Length on C, and find the Bung, Head, and Middle Diameters on D, noting the three Numbers opposite them on C; then to the Sum of the first and second add four times the third, and the Sum is the Measure sought. How to obtain the same in Gallons will be shewn presently.

To engage the Reader's Attention still more in favour of this Method, I shall shew what Theorems are thence deducible, for measuring the several Varieties of Casks, mention'd in the first *Scholium* of the last Chapter. For as the Forms of those Solids are given, we can by help of the Equation, defining the Curve by whose Revolution they may be supposed generated, find m the *Middle Diameter*, which in Practice, when we do not know the Form, must be found by actual Mensuration.

Cor. 5. For if the Sides of a Solid were streight, and the two Ends two Ellipses, parallel and similarly posited, or so that the longest Axis of the one was parallel to the shortest of the other, and B, W denoted them of the greatest End, also b, w those of the least respectively; then, if this Solid was cut through its Middle by a Plane, parallel to its Ends, the Section which is parallel to B, b, will be denoted by $\frac{B+b}{2}$, and that parallel to W, w, by $\frac{W+w}{2}$; whence the Area of the greater End is $B\times W\times a$, the lesser $b\times w\times a$, and of the Section in the Middle $\frac{B+b}{2}\times\frac{W+w}{2}\times a$, (as is evident from *Cor.* 1. *Prop.* 5. *Chap.* 3.) If these are put for S′, S‴, and S″, in *Cor.* 3. foregoing, we have $\overline{B\times W+b\times w+\overline{B+b}\times\overline{W+w}}\times\frac{al}{6}$

for

for the Meaſure of this Solid, or $\overline{W \times 2B+b+w \times 2b+B} \times \frac{al}{6}$, *which is a general Theorem for meaſuring all ſtreight-ſided Tuns, whoſe Ends are circular, elliptical or rectangular, and ſituate as above.* For let the two Ends be Circles, then WB and $w=b$; whence for meaſuring the Fruſtum of a right Line we have $\overline{2B^2+2b^2+2Bb} \times \frac{al}{6}$, or $\overline{B^2+b^2+Bb} \times \frac{al}{3}$, as was ſhewn *Cor.* 3. *Prop.* 2. *Chap.* 4.

If the two Ends are Squares, then a in the laſt Theorem muſt be made equal to Unity, whence $\overline{B^2+b^2+Bb} \times \frac{l}{b}$, is the Meaſure of a Tun in the Shape of the Fruſtum of a ſquare Pyramid.

But if the Ends are Rectangles, then $\overline{W \times 2B+b+w \times 2b+B} \times \frac{l}{6}$ is the Meaſure of all ſtreight-lined Tuns, whoſe Breadths at Top and Bottom are b, B, or B, b, and Widths correſponding w, W, or W, w.

We might from the ſame Principle deduce the Meaſures of Tuns, having their Ends of any other Figures, but they never occurring in Practice, I decline ſuch uſeleſs Purſuits, and proceed to

Cor. 6. If MMPP be *the Fruſtum of a parabolic Conoid*, the Diameter of whoſe greater End is b, leſſer h, and Length l, as before; alſo put p the Parameter of the generating Parabola; then (by *Cor.* 3. *Prop.* 2. *Chap.* 2.) $p \times l = \frac{b+h}{2} \times \frac{b-h}{2}$, and $p \times \frac{l}{2} = \frac{b+m}{2} \times \frac{b-m}{2}$, whence $m^2 = \frac{b^2+h^2}{2}$,

the Square of the middle Diameter; therefore, by *Cor.* 4. foregoing, $\overline{b^2+b^2+4m^2}\times\frac{al}{6}=\overline{b^2+b^2}\times\frac{al}{2}$, as found before, *Page* 123.

Cor. 7. If MMPP represent the *Frustum of a Spheroid*, whose transverse Axis is t, the rest as above, then (from *Cor.* 2. *Prop.* 1. *Chap.* 2.) $t^2 : b^2 :: t^2-4l^2 : b^2$, and $t^2 : b^2 :: t^2-l^2 : m^2$, whence $m^2=\frac{3b^2+b^2}{4}$, the Square of the Middle-Diameter, therefore now $\overline{b^2+b^2+4m^2}\times\frac{al}{6}=\overline{2b^2+b^2}\times\frac{al}{3}$, as was shewn *Page* 126.

These three Cases agree exactly with the Solutions given from other Principles, but in those that follow, especially the two last, which do not admit of a perfect Cubation; the Solutions found from this Method, will deviate a small matter from the Truth, but never so much as to be of the least Consequence in Gauging.

Cor. 8. Let MMPP be the *Frustum of a parabolic Spindle*, the Letters b, b, l, m, denoting the same Lines as before; then from what was said *Prop.* 5. *Chap.* 4. $b=b-\frac{2l^2}{p}$, and therefore $m=b-\frac{l^2}{2p}=\frac{3b+b}{4}$, *the Middle Diameter.* Put this for m (*Cor.* 4.) and we have $\overline{2b^2+b^2-\frac{3}{8}\times\overline{b-b}|^2}\times\frac{al}{3}$, too much by $\overline{b-b}|^2\times\frac{al}{120}$, which can never amount to $\frac{1}{10}$th of a Gallon in Gauging any Cask to be met with in Practice. But if we should suppose another Frustum exactly equal to this in every respect join'd to it; then we have the Value of five Equidistant Diameters of this Solid, the

the Extreme ones being b, b, the next to theſe m, m; and the middle one b; whence by the Rule againſt five Ordinates in the Table, we have $\overline{7A+32B+12C}\times\frac{al}{6}$, which is equivalent to $\overline{14b^2+\frac{32}{8}\times\overline{3b+b}^2+3b^2}\times\frac{al}{90}$; for $A=ab^2+ab^2=2ab^2$, $B=am^2+am^2=a\times\overline{\frac{3b+b}{8}}^2$, and $C=ab^2$: which put for A, B, C, gives the Expreſſion above; and that contracted, is equal to $\overline{2b^2+b^2-\frac{2}{5}\times\overline{b-b}^2}\times\frac{al}{3}$, the Meaſure of a parabolic Spindle, the Diameter of whoſe Ends are b, that in the Middle betwixt them b, and Length l; exactly the ſame as was ſhewn *Prop.* 5. *Chap.* 4.

Cor. 9. Laſtly, let MMPP repreſent the *Fruſtum of a Circular*, or *Equilateral Hyperbolic Spindle*, and t the Diameter of the generating Curve, then from what was ſaid *Prop.* 6. *Chap.* 4. we have for the $\left\{\begin{matrix}\textit{Firſt}\\ \textit{Second}\end{matrix}\right\}$

$$t=\left\{\begin{matrix}\frac{1}{2}\times\frac{l^2}{b-b}-\frac{1}{2}\times\overline{b+b}+b\\ \frac{1}{2}\times\frac{l^2}{b-b}+\frac{1}{2}\times\overline{b+b}-b\end{matrix}\right\}\text{ and }m=\left\{\begin{matrix}b-t+\sqrt{t^2-\frac{l^2}{4}}\\ b+t-\sqrt{t^2+\frac{l^2}{4}}\end{matrix}\right.$$

But ſince in the Theorems (*Scholium* 2. *Ch.* 4.) for meaſuring theſe Solids, there is S a Segment of the generating Curve, therefore we can no otherwiſe ſhew the Coincidence of the Meaſure deduced from this Method, with the true Solution there given, but by a particular Example; for which purpoſe let $b=32$, $b=24$, and $l=40$, the ſame as in the above Scholium, then for the

$$\left.\begin{array}{l}\textit{Circle}\\ \textit{Eq. Hyp.}\end{array}\right\} t = \left\{\begin{array}{c}104\\ 96\end{array}\right\} m = \left\{\begin{array}{l}30{,}059\\ 29{,}939\end{array}\right.$$

And therefore (by the Theorem *Corollary* 4.) $\overline{b^2+b^2+4m^2} \times \frac{al}{6}$ gives for the

$$\begin{array}{lcccc} & & \textit{C. Inches.} & & \textit{A.Gall.} \\ \left.\begin{array}{l}\textit{Circle}\\ \textit{Eq.Hyp.}\end{array}\right\} & = & \left\{\begin{array}{l}27301{,}12\\ 27150{,}34\end{array}\right\} & \text{or} & \left\{\begin{array}{l}96{,}81\\ 96{,}28\end{array}\right.\end{array}$$

The first differs but $\frac{5}{100} = \frac{1}{20}$th Parts of a Gallon from the Truth; and the Second but $\frac{6}{100} = \frac{1}{16\frac{2}{3}}$ds of a Gallon.

From what has been said, it plainly appears by this Method, you are certain either to have the true Content, or within the 20th Part of a Gallon; whereas I may venture to affirm, by the common way of guessing at the Form of a Cask, and thence finding its Content (by reducing it to a Cylinder from the ordinary Multipliers) you can never be sure within two or three Gallons, or perhaps more; and this Method being also very ready in Practice, I would fain hope some time or other to see it generally follow'd by the Officers of Excise.

Having now laid down all the necessary Theorems for Gauging, I shall proceed to shew their Application to Practice.

PART

PART III.

The Practice of Gauging.

IF what has been faid in the preceding Pages was thoroughly confidered, there would be no neceffity of adding any thing further; for fince

A	*Gallon of Ale*	contains	282	*Cub. Inch.*
A	*Gallon of Wine*	contains	231	*Cub. Inch.*
A	*Bufhel of Malt*	contains	2150,42	*Cub. Inch.*

'Tis manifeft if the Meafure of any Solid in Cubic Inches, found by the foregoing Rules, be divided by any of the Numbers above, the Quotient will refpectively fhew, how many Gallons of Ale, Wine, or Bufhels of Malt, a Veffel in that Form will contain.

But as this Tract may fall into the Hands of fome, whofe Employment requires they fhould be acquainted with the Practice of Gauging, and yet have not Time, or perhaps Patience to inform themfelves with the Theory, and fo might have fome Difficulty to fix on the proper Rules or Theorems for their purpofe, as they are intermix'd with the inventive Part, and alfo the Theorems themfelves, require fome farther elucidation to render them more eafy to Learners; I have for their fakes added this Third Part, which contains the Practice, or all the Rules for Gauging, neceffary to be known by every Officer in the Excife.

CHAP. I.

Of Cask-Gauging.

THIS Chapter is divided into three Sections, containing three different Methods for Gauging a full Cask. The *first* and *second* are on Supposition that the Form of the Cask is known, and also that it is generated by a Circle, or one of the Conic Sections turning round its Axis, or an Ordinate thereof. But the *third Method* is by taking a *fourth Dimension* without any regard to the Form, which is required in the other two.

The Number of Forms into which Casks are commonly distinguish'd in Practice, are only three, *viz.* two Frustums of a Spheroid, two Frustums of a Parabolic Spindle, and two Frustums of a Cone; but as there undoubtedly may be others betwixt these, and also because we have given Rules for Measuring some of the intermediate ones, we distinguish them in six Forms; the Order and Names of which are as follows:

Variety the	1	*Two equal Frustums of a*	*Spheroid.*
	2		*Circular Spindle.*
	3		*Parabolic Spindle.*
	4		*Eqilat. Hyp. Spin.*
	5		*Parabolic Conoid.*
	6		*Cone.*

Some have laid down Rules to discover these Forms one from another by Inspection, but as I conceive that impossible, instead thereof we refer the Reader to the Figures 58, 59, 60, 61, 62, 63, which are exactly drawn for the several Varieties above, when the Bung-Diameter is 32, Head

Head 26, and Length 40 Inches; by which he will acquire a better Idea of the ſeveral Varieties above, than can poſſibly be convey'd by any verbal Deſcription whatever.

SECTION I.

PROP. I.

The Length, Head, and Bung-Diameters of a Cask in the Shape of two equal Fruſtums of a Spheroid *being given, to find its Meaſure in Ale or Wine-Gallons.*

Solution. To twice the Square of the Bung-Diameter, add the Square of that of the Head, and multiply the Sum by the Length of the Caſk; then if that Product is multiplied

by $\left\{\begin{array}{l},0009284\\,0011333\end{array}\right\}$ or divided by $\left\{\begin{array}{r}1077,15\\882,35\end{array}\right\}$ for $\left\{\begin{array}{l}\textit{Ale,}\\\textit{Wine,}\end{array}\right.$

the Product, or Quotient, will be the Meaſure requir'd.

Or by the Sliding-Rule.

Set $\left\{\begin{array}{l}32,82\\29,7\end{array}\right\}$ for $\left\{\begin{array}{l}\textit{Ale}\\\textit{Wine}\end{array}\right\}$ on D, to the Caſk's Length on C, and on D find both the Bung and Head-Diameters, noting the two Numbers oppoſite them on C; then if the ſecond is added to twice the firſt, the Sum will be the Meaſure required.

Example. Let ABCD (*Fig.* 58.) repreſent a Caſk in the Shape of two equal Fruſtums of a Spheroid, whoſe Bung-Diameter EF is 32 Inches, Head BD=AC 26 Inches, Length GH 40 Inches, and its Meaſure requir'd in Ale and Wine Gallons.

Twice the Square of (32) the Bung- - - - - -	Diam. is	2048
The Square of (26) the Head-		676
Their Sum - - - - -		2724
Multiplied by (40) the Length -		40
Gives - - - - -		108960
This multiplied by ,0009284 *inverted*		4829000,0
		9806
		218
		87
		4
Gives the Content in *Ale Gallons* -		101,15
Or, - - - - - -		108960
Multiplied by		3331100,0
		10896
		1090
		327
		33
		3
Gives the Content in *Wine Gallons*		123,49

Or, By the Sliding-Rule, I ſet 32,82 on D, to 40 on C, then againſt 32 and 26 on D, is 38,03 and 25,1 on C; therefore by the Rule, twice the firſt (38,03) *viz.* 76,06, added to 25,1 the ſecond, gives 101,16 *Ale Gallons.*

But 29,7 on D being ſet to 40 on C, then againſt 32 and 26 on D, is 46,4, and 30,7 on C; therefore twice (46,4) the firſt, *viz.* 92,8, added to 30,7, gives 123,5 *Wine Gallons*, both nearly the ſame as before.

PROP.

PROP. II.

The Length, Bung, and Head-Diameters of a Cask in the Shape of two equal Frustums of a Circular Spindle *being given, to find its Measure in Gallons.*

Solution. To twice the Square of the Bung-Diameter, add the Square of the Head-Diameter, and let the Sum be called the *first Number*; then to the Square of the Length, add the Square of the Head-Diameter, and from the Sum take the Square of the Bung-Diameter, the Remainder multiplied by fourteen times the Square of the Difference of the Diameters is *a Dividend*: And thirty-five times the Square of the Length, added to five times the Square of the Difference of the Diameters, is the Divisor: the Quotient of this Division is the *Second Number*: Then if the Difference of the first and second Numbers be multiplied by the Length of the Cask, and that Product

$$\text{by}\left\{\begin{array}{l},0009284\\,0011333\end{array}\right\}\text{or divi-}\left\{\begin{array}{c}1077,15\\882,35\end{array}\right\}\text{for}\left\{\begin{array}{l}\text{Ale,}\\\text{Wine,}\end{array}\right.$$
$$\text{ded by}$$

the Product, or Quotient, will be the Measure required.

Or, By the Sliding-Rule,

$$\text{Set}\left\{\begin{array}{l}32,82\\29,7\end{array}\right\}\text{for}\left\{\begin{array}{l}\text{Ale}\\\text{Wine}\end{array}\right\}\text{on D, to the Cask's}$$

Length on C, and find both the Bung and Head-Diameters, as also their Difference on D, noting the three Numbers opposite them on C; then if to the second twice the first be added, and $\frac{3}{10}$ths of the third taken from the Sum, the Remainder will be *nearly* the Measure sought.

Example. Let the Dimensions of a Cask ABCD (*Fig.* 59) representing two equal Frustums of a circular Spindle, be the same as in the preceding Proposition.

OPERA-

OPERATION.

By the laſt, the *firſt Number* is -	2724
The Square of (40) the Length is -	1600
The Square of (26) the Head-Diam. is	676
Their Sum - - - - -	2276
From which take the Square of (32)	1024
There remains - - - - -	1252
Which multiply by ($14 \times 6 \times 6 =$)	504
	5008
	6260
The Product is the *Dividend*= -	631008
And 35 times the Square of 40 =	56000
To which 5 times 6×6 being added=	180
The Sum is the *Diviſor* - -	56180
Then 56180)631008(= -	11,232
Which taken from the 1ſt N°. leaves	2712,768
This multiplied by the Length -	40
Gives - - - - - -	108510,720
Which multip. by ,0009284 invert.	4829000,0
	97660
	2170
	868
	43
Gives the Content in *Ale Gallons*	100,741

Or,

Or, - - -	108510,720
Multiplied by ,0011333 inverted	33331100,0
	108511
	10851
	3255
	326
	33
	3
Gives the Content in *Wine Gallons*	122,979

By the Sliding-Rule, I set 32,82 on D to 40 on C, then against 32,26 and 6 on D, is 38,03, 25,1 and 1,34 on C; therefore 76,06 (=38,03 × 2) + 25,1—,4 (=1,34 ×, 3) gives 100,76 *Ale Gallons.*

And if 29,7 on D was set to 40 on C, then against 32,26 and 6 on D, is 46,42, 30,64 and 1,63 on C; therefore 92,84 (=46,42 × 2)+30,64 —,49 (=1,63 × 3) gives 122,99 *Wine Gallons*; both nearly the same as were found before by Computation.

Prop. III.

The Length, Bung, and Head-Diameters of a Cask (Fig. 61.) *representing two equal Frustums of a* Parabolic Spindle *being given, to find its Measure in Gallons.*

Solution. To twice the Square of the Bung-Diameter, add the Square of the Head Diameter, and from the Sum subtract $\frac{2}{5}$ths of the Square of the Difference of the Diameters; then if the Remainder be multiplied by the Cask's Length, and that Product

for

for $\left\{\begin{matrix}\text{Ale}\\ \text{Wine}\end{matrix}\right\}$ by $\left\{\begin{matrix},0009284\\ ,0011333\end{matrix}\right\}$ or divid. by $\left\{\begin{matrix}1077,15\\ 882,35\end{matrix}\right\}$

The Product, or Quotient, will be the Measure sought.

By the Sliding-Rule, set $\left\{\begin{matrix}32,82\\ 29,7\end{matrix}\right\}$ for $\left\{\begin{matrix}\text{Ale}\\ \text{Wine}\end{matrix}\right\}$ on D, to the Length on C, and find both the Bung and Head-Diameters, as also their Difference on D, noting the three Numbers opposite them on C; then if the second is added to twice the first, and $\frac{2}{5}$ths of the third taken from the Sum, the Remainder will be the Measure of the Cask proposed.

For an *Example* to this Proposition, let the Dimensions of the Cask be the same as those before.

Then twice the Square of the Bung added to that of the Head-Diam.	= 2724
$\frac{2}{5}$ths of (6x6) the Sq. of their Differ. is	14,4
Which taken from the first, leaves	2709,6
This multiplied by the Length,	40
Gives - - -	108384,0
And this mult. by ,0009284 invert.	4829000,0
	97546
	2168
	867
	43
Gives the Content in *Ale Gallons* =	100,624

Or,

Or,	108384,000
Multiplied by ,0011333 inverted	33331100,0
	108384
	10838
	3252
	325
	33
	3
Gives the Content in *Wine Gallons* =	122,835

But, when 32,82 on D, is ſet to 40 on C, then againſt 32,26 and 6 on D, is 38,03; 25,1 and 1,34 on C: therefore $38{,}03 \times 2 + 51{,}1 - 1{,}34 \times {,}4 = 100{,}62$ *Ale Gallons.*

And when 29,7 on D, is ſet to 40 on C, then on C againſt 32,26, and 6 on D is 46,42 30,64 and 1,63; hence $46{,}42 \times 2 + 30{,}64 - {,}4 \times 1{,}63 = 122{,}83$ *Wine Gallons.*

PROP. IV.

The Length, Bung and Head-Diameters of a Cask, (Fig. 62.) *repreſenting two equal Fruſtums of an equilateral* Hyperbolic Spindle *being given, to find its Meaſure in Gallons.*

Solution. Let twice the Square of the Bung-Diameter, added to that of the Head, be call'd the *firſt Number*; and to the Square of the Length add the Difference of the Squares of the Bung and Head-Diameters, let the Product of this by fourteen times the Square of the Difference of the Diameters, be *a Dividend*; alſo from 35 times the Square of the Length, take five times the Square of the Difference of the Diameters, the Remainder is *the Diviſor*; the Quotient of this Diviſion is

is the *second Number.* And the Difference betwixt the first and second Numbers, multiplied by the Length of the Cask, and that Product for $\left\{\begin{array}{l}\text{Ale}\\\text{Wine}\end{array}\right\}$ by $\left\{\begin{array}{l},0009284\\,0011333\end{array}\right\}$ or divided by $\left\{\begin{array}{r}1077,15\\882,35\end{array}\right\}$ the Product, or Quotient, will be the Measure sought.

By the Sliding-Rule; for $\left\{\begin{array}{l}\text{Ale}\\\text{Wine}\end{array}\right\}$ set $\left\{\begin{array}{r}32,82\\29,7\end{array}\right\}$ on D, to the Length on C, and find the Bung and Head-Diameters, as also their Difference on D, noting the three Numbers opposite them on C: Then if from the Sum of twice the first and second $\frac{1}{3}$ the third be taken, the Remainder is the Measure requir'd.

Example. Let the Dimensions of a Cask representing two equal Frustums of an equilateral hyperbolic Spindle be the same as before, and its Measure required in Ale, and Wine Gallons.

OPERATION.

The *first Number* is - - -	2724
The Square of (40) the Length =	1600
The Square of (32) the Bung-Diam. =	1024
Their Sum - - - - -	2624
From which take the Square of 26 =	676
There Remains - - - -	1948
This multiplied by 14×6×6 =	504
	7792
	9740
Gives *the Dividend* =	981792

From

From 35 times the Square of 40 =	56000
Take 5 times the Square of 6 =	180
The Remainder is the *Divisor* =	55820
Therefore the 2*d Numb.* is $\frac{981792}{55820}$ =	17,59
Which taken from the 1ſt N°. gives	2706,41
This multiplied by the Length	40
Gives - - - -	108256,40
And this by ,0009284 inverted	4829000,0
	97431
	2165
	866
	43
Gives the Content in *Ale Gallons*	100,505
Or,	108256,4
Multiplied by ,0011333 inverted	33331100,0
	108256
	10826
	3248
	325
	32
	3
Gives the Content in *Wine Gallons*	122,690

But, *by the Sliding-Rule,* when 32,82 on D, is ſet to 40 on C; then againſt 32,26 and 6 on D, is 38,03, 25,1 and 1,34 on C; therefore 2 x 38,03 + 25,1 — $\frac{1}{2}$ x 1,34 = 100,49 *Ale Gallons.*

Or,

Or, 29,7 on D, set to 40 on C, against 32,26 and 6 on D is 46,42 30,7 and 1,63 on C, therefore $2 \times 46{,}42 + 30{,}7 - \frac{1}{2} \times 1{,}63 = 122{,}7$ *Wine Gallons*, both nearly the same as before.

Prop. V.

The Length, Bung, and Head-Diameters of a Cask, (Fig. 63.) *in the Shape of two equal Frustums of a* Parabolic Conoid *being given, to find its Measure in Gallons.*

Solution. To the Square of the Bung-Diameter, add that of the Head, multiply the Sum by the Length of the Cask, and this Product

$$\text{for}\begin{cases}\text{Ale}\\\text{Wine}\end{cases}\text{by}\begin{cases},0013925\\,0017\end{cases}\text{or divided by}\begin{cases}718{,}1\\588{,}23\end{cases}$$

the Product, or Quotient, will be the Measure required.

By the Sliding-Rule: for $\begin{cases}\text{Ale}\\\text{Wine}\end{cases}$ set $\begin{cases}26{,}8\\24{,}25\end{cases}$ on D to the Cask's Length on C, and find the Bung and Head Diameters on D, noting the two Numbers opposite them on C, whose Sum will be the Measure sought.

Thus in our Example, The Square

$\begin{cases}32\\26\end{cases}$ the $\begin{cases}\text{Bung}\\\text{Head}\end{cases}$ Diam. is	1024 676
Their Sum - -	1700
Multiplied by (40) the Length	40
Gives - - - -	68000
And this multiplied by ,0013925	0,0013925
	111400000 83550
Gives the Content in *Ale Gallons*	94,6900000

Or,

Or,	68000
Multiplied by	0,0017
	476000
	68
Gives the Content in *Wine Gallons*	115,6000

By *the Sliding-Rule*, when 26,8 on D, is set to 40 on C, then against 32 and 26 on D, is 57,1 and 37,6: whose Sum is 94,7 *Ale Gallons*.

But 24,25 on D, set to 40 on C, then against 32 and 26 on D, is 69,63 and 45,97: whose Sum is 115,6 *Wine Gallons*, as before.

PROP. VI.

The Length, Bung, and Head-Diameters of a Cask (Fig. 64.) *representing two equal* Frustums *of an upright* Cone, *being given, to find the Measure in Gallous.*

Solution. Multiply the Sum of the Bung and Head-Diameters by the latter, and to the Product add the Square of the former; that Sum multiplied by the Length of the Cask, and this

for $\left\{\begin{matrix}\text{Ale}\\\text{Wine}\end{matrix}\right\}$ by $\left\{\begin{matrix},0009284\\,0011333\end{matrix}\right\}$ or divided by $\left\{\begin{matrix}1077,15\\882,35\end{matrix}\right.$

the Product, or Quotient, will respectively be the Measure in Ale or Wine-Gallons, as required.

By *the Sliding-Rule*; for $\left\{\begin{matrix}\text{Ale}\\\text{Wine}\end{matrix}\right\}$ set $\left\{\begin{matrix}65,64\\59,4\end{matrix}\right\}$ on D to the Cask's Length on C; and on D find both the Sum, and Difference of the Diameters, noting the two Numbers opposite them on C; then if the second is added to three times the first, four times the Sum is the Measure sought.

Example. Let the Dimensions of this Cask be the same with the foregoing.

OPERATION.

The Bung } Diameter is {	32
The Head } Diameter is {	26
Their Sum	58
Multiplied by	26
	348
	116
Gives	1508
To which add the Square of 32 =	1024
The Sum is	2532
This multiplied by the Length	40
Gives	101280
And this by ,0009284 inverted,	4829000,0
	91152
	2026
	810
	41
The Prod. is the Content in *Ale Gall.*	94,029
Or,	101280
Multiplied by ,0011333 inverted	33331100,0
	101280
	10128
	3038
	304
	30
	3
Gives the Content in *Wine Gall.*	114,783

But,

But, by *the Sliding-Rule*, when 65,64 on D, is set to 40 on C; then against 58 (the Sum of the Diameters) and 6 (their Difference) on D, is 31,23 and ,34 on C respectively; therefore if to three times the first, *viz.* 93,69, the second ,34 be added, we have 94,03 for the Measure in *Ale Gallons*.

Or, 59,4 on D set to 40 on C, then against 58 and 6 on D is 38,12, and ,41 on C; therefore ,41 added to three times the first, *viz.* 114,36, gives 114,77 *Wine Gallons*, nearly the same as found by Computation.

We might have set $\left\{ \begin{matrix} 32,82 \\ 29,7 \end{matrix} \right\}$ for $\left\{ \begin{matrix} \text{Ale} \\ \text{Wine} \end{matrix} \right\}$ on D, to the Cask's Length on C; then if the Sum of the Diameters, and their Difference were found on D, three fourths of the first, added to one fourth of the second, would have been the Measure of the Cask proposed.

SCHOLIUM.

Thus we have done with the first thing proposed in this Chapter, wherein, besides the Theorems for computing the Measures of the several Varieties of Casks mentioned at the beginning, we have given the Method of Gauging, each *by one Setting of the Lines* C,D *on the Sliding-Rule*, four of which are strictly true, and the others (*viz.* those for the *Circular* and *Hyperbolic Spindle*) so near, that they will never differ from the Truth, by the $\frac{1}{20}$th of a Gallon, in gauging any Cask ever to be met with in Practice. And since the Operation is so very easy, I am surprized this Method was never practised; because I am very certain, any of the abovementioned Casks may be gauged with half the trouble required in the common Method of reducing them to Cylinders

with this Advantage, that here you are ever *sure of the true Content*, if the Cask has the assigned Form; whereas by any Rules hitherto publish'd for finding the *Mean Diameters*, it is doubtful.

And to render this Method still more easy for Practice, I would advise every Officer that intends to make use of it, to have the Points answering the Numbers 32,28 and 29,7 on D, mark'd with a Brass Pin, and A, W plac'd against them respectively, to denote Ale and Wine; which Points, in this way of Gauging, might be call'd *Gauge Points*, as 18,94 and 17,14 are in the common Method.

However, that this Treatise may not appear deficient, we shall now proceed to the second thing proposed, *viz.*

SECTION II.

The Manner of Gauging a full Cask by the Mean Diameter.

PROP. I.

The Bung and Head-Diameters of a Cask, resembling the Frustum of a Spheroid, *being given, to find the* Mean Diameter.

Solution. To the Head-Diameter add two thirds of the Difference of the Diameters, and to this add one third of the Quotient arising from dividing the Square of the Difference of the Diameters, by the Sum of the Head and twice the Bung-Diameters, so will you *nearly* have the Mean required.

Example. Let the Bung be 32, and Head 26 Inches, (as in the last *Section*) then we have this

OPERATION.

OPERATION.

The Head-Diameter is -	26
$\frac{2}{3}$ds of 6 the Diff. of the Diam. =	4
$\frac{1}{3}$ of $\frac{36}{90} = \frac{12}{90} = \frac{1,2}{90} =$ -	0,1333
Their Sum is the Mean, *viz.*	30,1333

PROP. II.

The Bung and Head-Diameters of a Cask, resembling the Frustum of a Circular Spindle, *being given, to find the Mean Diameter.*

Solution. To the Head-Diameter add two thirds of the Difference of the Diameters, and to this eleven sixtieths ($\frac{11}{60}$) of the Square of the Difference of the Diameters, divided by the Head and twice the Bung-Diameter, the Sum will be the mean sought, *nearly*.

Thus in this Example;

The Head-Diameter is - -	26
$\frac{2}{3}$ of 6 - -	4
$\frac{11}{60}$ of $\frac{36}{90} = \frac{11 \times 6}{900} = \frac{,66}{9} =$ -	0,0733
Their Sum is the Mean, *viz.*	30,0733

PROP. III.

The Bung and Head-Diameters of a Cask, resembling the Frustum of a Parabolic Spindle, *given, to find the Mean Diameter.*

Solution. To the Head-Diameter add two thirds of the Difference of the Diameters, and to this four thirtieths ($\frac{4}{30} = \frac{4}{3}$) of the Quotient arising from dividing the Square of the Difference of the Diameters by the Head and twice the Bung-Diameter,

 the

the Sum of these three Terms is the Mean sought *nearly*.

The Head-Diameter is - -	26
$\frac{2}{3}$ of 6 = - -	4
$\frac{4}{30} \times \frac{36}{90} = \frac{4 \times 12}{900} = \frac{,48}{9} =$	0,0533
Their Sum is the Mean, *viz.*	30,0533

Prop. IV.

The Bung and Head-Diameters of a Cask, representing the Frustum of an Equilateral Hyperbolic Spindle, *given, to find the Mean Diameter.*

Solution. To the Head-Diameter add two thirds of the Difference of the Diameters, and to this add one twelfth ($\frac{1}{12}$) of the Quotient found by dividing the Square of the Difference of the Diameters by the Head added to twice the Bung-Diameter, the Sum is nearly the Mean sought.

The Head-Diameter -	26
$\frac{2}{3}$ of 6 = - - -	4
$\frac{1}{12}$ of $\frac{36}{90} = \frac{3}{90} = \frac{,3}{9} =$ -	0,0333
The Sum is the Mean, *viz.*	30,0333

Prop. V.

The Bung and Head-Diameters of a Cask, in the Shape of the Frustum of a Parabolic Conoid, given, to find the Mean Diameter.

Solution. To the Head-Diameter add half the Difference of the Diameters, and to this one fourth of the Square of the said Difference, divided by the Sum of the Diameters, the Sum is the Mean sought, *nearly*.

The

29) 4,5 (,1552	The Head-Diam. is	26
160	$\frac{1}{2}$ of 6 = -	3
150	$\frac{1}{4}$ of $\frac{36}{58} = \frac{1}{4} \times \frac{18}{29} = \frac{1}{2} \times \frac{9}{29} =$	0,1552
50	The Sum is the Mean, *viz.*	29,1552

PROP. VI.

The Bung and Head-Diameters of a Cask, resembling the Frustum of an Upright Cone, *given, to find the Mean Diameter.*

Solution. To the Head-Diameter add half the Difference of the Diameters, and to this one twelfth ($\frac{1}{12}$) of the Quotient found by dividing the Square of the said Difference, by the Sum of the Diameters, the Sum is nearly the Mean requir'd.

58) 300 (,0517	The Head-Diam. is	26
100	$\frac{1}{2} \times 6 =$	3
420	$\frac{1}{12} \times \frac{36}{58} = \frac{3}{58}$	0,0517
	Their Sum is the Mean, *viz.*	29,0517

Having shewn how to find a Mean Diameter for any of the six Varieties, (*Page* 198.) that is to find the Diameter of a Cylinder of equal Length and Magnitude to a Cask propos'd, which is deduc'd from the Theorems (*Page* 165.) we must now shew what Use is to be made of it when obtain'd: and as I suppose the chief Design of reducing Casks to Cylinders, was from a Supposition that their Contents might thence be more easily computed by the Sliding-Rule, than when consider'd in their original Forms, (tho' we have shewn it is otherwise) so here we shall only give the Method of finding the Content after that manner, the numeral Operation being easy from what was said *Prop.* 1. *Chap.* IV. *Part.* 2.

Prop. VII.

The Length, Bung, and Head-Diameters of a Cask, representing any of the six Varieties, (Page 198.) *given, to find its Measure in Gallons by the Sliding-Rule.*

Solution. First find the mean Diameter by one of the preceding Propositions, proper to the Cask proposed.

Then for $\left\{\begin{array}{l}\text{Ale}\\ \text{Wine}\end{array}\right\}$ set $\left\{\begin{array}{l}18{,}94\\ 17{,}14\end{array}\right\}$ on D, to the Cask's Length on C, and on C, opposite the mean Diameter on D, will be the Measure in Gallons of the Casks proposed.

Example. Let the Dimensions of the several Varieties of Casks be the same with those in the preceding Propositions, we have shewn the Mean Diameters for them to be as hereunder, *viz.*

$$\text{For the Frustum of a}\left\{\begin{array}{l}\textit{Spheroid}\\ \textit{Cir. Spin.}\\ \textit{Par. Spin.}\\ \textit{Equil. H. S.}\\ \textit{Par. Con.}\\ \textit{Cone}\end{array}\right\}\text{the Mean Diam. is}\left\{\begin{array}{l}30{,}1333\\ 30{,}0733\\ 30{,}0533\\ 30{,}0333\\ 29{,}1552\\ 29{,}0517\end{array}\right.$$

Therefore by the Rule above,

$$\begin{array}{l}\text{I set } 18{,}94\\ \text{on D, to } 40\\ \text{on C,}\end{array}\left\{\text{then against}\left\{\begin{array}{l}30{,}1333\\ 30{,}0733\\ 30{,}0533\\ 30{,}0333\\ 29{,}1552\\ 29{,}0517\end{array}\right\}\text{on D is}\left\{\begin{array}{l}101{,}2\\ 100{,}7\\ 100{,}6\\ 100{,}5\\ 94{,}7\\ 94{,}0\end{array}\right\}\text{respectively on C.}\right.$$

Which are the Contents in Ale Gallons of the above Casks, and nearly the same as was found before in the last Section.

Scho-

SCHOLIUM.

The above Rules for finding the Mean Diameters are not only very near the Truth, but alſo very eaſy in Practice; however, they require a ſmall Multiplication and Diviſion, which is more than many care to be at the trouble of, rather chuſing to follow Methods leſs exact, that can be remembred, and ſo the Mean Diameter found as it were mentally: And therefore we muſt in this place ſet down ſome fixt Multipliers, which will ſerve pretty well, if (as commonly happens) the Bung-Diameter be about four or five times the Difference betwixt it and the Head.

They are thus for the ſeveral Varieties foregoing, *viz.*

For two Fruſtums of a	*Spheroid*	,69
	Circular Spindle	,68
	Parab. Spindle	,67$\frac{1}{2}$
	Equil. Hyp. Spindle	,67
	Parab. Conoid	,52$\frac{1}{2}$
	Cone	,51

By which if the Difference betwixt the Bung and Head-Diameters of a Caſk reſembling any of the above Solids be multiplied, and the Product added to the Head-Diameter, the Sum will nearly give the Mean thereof.

The Methods deliver'd in this and the laſt Section would be ſufficient for Caſk-Gauging, as being each of them very exact, and (which indeed is chiefly to be conſider'd) ready in Practice, if there was not an Objection to both, which is too material to paſs over in ſilence: It is this; In each of them the *Shape* or Figure of the Caſk is ſuppoſed given, or eaſily determined by Inſpection; but this I conceive to be the greateſt Difficulty of all,

all, and is generally thought so by those who have been most conversant therein: nay, if it was otherwise, why have so many Mathematicians, even those of the first Rank, taken so much pains to give us Methods for determining the Loci describ'd by Lines or Angles, moving according to some certain Laws? If it was so easy a matter to determine the Curve, (by whose Revolution on some Line as an Axis, any given Solid may be suppos'd generated) meerly by Inspection? For in what I have just mention'd, you not only have an occular View of the Curve, but also the Law by which it is describ'd; therefore we must have recourse to some other Methods, which are built on a more certain Foundation, of which I think the following one (deduc'd from the last *Chapter* of *Part* 2.) is better fitted than any, or indeed the best the Nature of the thing admits of, whether we regard its Accuracy or Readiness in Practice; but as therein a fourth Dimension is requir'd, *viz.* one more than in the other two, we must shew an easy Method of taking that at the same time the Length is measur'd; and this is the third thing proposed in this Chapter.

SECTION III.

Of Gauging a full Cask by help of a Fourth Dimension, *without knowing to which Variety it belongs.*

To render this Affair as clear as possible, let AA'DD' (*Fig.* 66.) represent a Cask, whose Head-Diameters are AD, A'D', and Bung EF; also let the Center of each End be at O, O': then if a streight Ruler, having a Pin or something of that kind in its Middle, which is put in the Bung-Hole at E, be turn'd about the same as a Center,

Center, till a Plumb-Line let fall from either end would pass by O the Center thereof, and fixed in that Position; then two streight Rulers being apply'd to each End, so as passing by O, O', the Center of both, they would also pass by the Ruler mention'd above, which we will denote by P P', or *p p'*; 'tis plain from this Position of the Rulers if the Sum of the Parts AP, A'P', or A*p*, A'*p*, which are easily measured, be taken from EF the Bung Diameter, you'll have AD, or A'D' those of the Head. And the Rulers remaining as before, if the Points M, M', or *m*, *m'*, be taken in the middle betwixt the Points P, P', or *p*, *p'*, and any common Foot-Ruler be apply'd successively to M, M', or *m*, *m'*, in a Position as nearly parallel to DP, DP, as possible, and slid down till it meet the Cask, suppose in N, N', you'll have the Measures of the Parts MN, M'N', whose Sum being taken from EF, the Bung-Diameter, will leave NR, or N'R', that in the Middle betwixt the Bung and Head: And lastly, if the Ends of the Ruler PP', which is fixt in the Bung, be moved in a vertical Position round the Pin at E (which is done by letting it constantly touch the Rulers DP, D'P') till the Part A'P' be equal to AP, then the Distance betwixt the Intersections of this Rule with those applied to each End, (as before directed) will be the Length of the Cask. And I must further observe, both this Length and consequently the Middle-Points M, M' would be obtain'd very readily, if the Rule PP' was divided into Inches, and number'd both ways from E its Middle. But I shall not take upon me to give a particular Description, how this Rule for measuring the Lengths of Casks, and finding their middle Diameters, may be most conveniently made, that belonging to the Instrument-Makers; but I am persuaded

ſuaded a better Contrivance than the Callipers might be found, which would render the following Method of Gauging vaſtly eaſy in Practice; and for its Truth, I will preſume to ſay, 'tis much more ſo, than any that has yet been offer'd for that purpoſe: for the Confirmation of which, I ſubmit to any one who will only take the trouble to gauge any Caſk by it, and after fill it with Water carefully meaſured.

The General PROPOSITION.

The Length, Bung, and Head-Diameters of a Caſk, as alſo another Diameter taken in the Middle betwixt them, being given, to find its Meaſure in Gallons.

Solution. To the Square of the Bung-Diameter add that of the Head, and four times the Square of the middle Diameter; let this Sum be multiplied by the Length of the Caſk, and that Product for

$$\left.\begin{matrix}\text{Ale}\\ \text{Wine}\end{matrix}\right\} \text{by} \left\{\begin{matrix},0004642\\ ,0005667\end{matrix}\right\} \text{or divi-}\ \text{ded by} \left\{\begin{matrix}2154,32\\ 1764,71\end{matrix}\right.$$

the Product, or Quotient, will be the Meaſure of the Caſk propoſed.

By the Sliding-Rule; For $\left\{\begin{matrix}\text{Ale}\\ \text{Wine}\end{matrix}\right\}$ ſet $\left\{\begin{matrix}46,4\\ 42,\end{matrix}\right\}$ on D, to the Caſk's Length on C, and find both the Bung, Head, and Middle Diameters on D, noting the three Numbers oppoſite them on C; then if to the Sum of the firſt and ſecond, four times the third be added, you'll have the Meaſure required.

One Example will be sufficient to illustrate this.

Let the Length OO′ (*Fig.* 66.) of the Cask AA′DD′ be 40 Inches,

its $\left\{\begin{array}{l}\text{Bung}\\ \text{Head}\\ \text{Middle}\end{array}\right\}$ Diameter $\left\{\begin{array}{l}32=EF\\ 26=AD\\ 30{,}4=NR\end{array}\right\}$ and its Measure required in Gallons;

The Sq. of 32 the Bung Diam.	=	1024
The Sq. of 26 the Head Diam.	=	676
4 times the Sq. of 30,4 the Mid. Diam.	=	3696,64
Their Sum	=	5396,64
Multiplied by the Length	=	40
		215865,60
And this by ,0004642 inverted		2464000,0
		86346
		12952
		863
		43
Gives the Content in Ale Gallons	=	100,204

Or, *By the Sliding-Rule*; I set 46,4 on D, to 40 on C, then against 32, 26, and 30,4 on D, is 12,8, 19 and 17, 1 on C, respectively; therefore 12,8+19+ (4 × 17,1) 68,4=100,2 as found before by Computation; and after the same manner by using the Numbers proper for Wine Measure, the Content will be found 122,3 Wine-Gallons.

CHAP.

CHAP. II.

Of ULLAGING.

AS the *Positions* of a Cask for the Convenience of drawing off, may with respect to its Axis (which is a Line suppos'd drawn from the Center of one End to that of the other) be either *upright* or *horizontal*, so from thence there will naturally be two Cases in finding the Quantity of Liquor drawn out or remaining in a Cask not quite full, and called *its Ullage*, which are very different from each other. To find the Ullage of an *horizontal* or *lying* Cask, is a very difficult Problem, except for two of the Varieties, *viz.* the 5th and 6th; the first of these may be solved by *Prop.* 3. *Chap.* 4. *Part.* 2. and the second from *Corol.* 5. *Prop.* 1. *Chap.* 5. of the same Part; also by the Theorem *Pag.* 147. you may nearly find the Ullage of any of the other Varieties; but they are all too tedious for Practice, so we must have recourse to other Methods, of which that now generally followed, is by help of the Lines mark'd S L, S S, on the Sliding-Rule.

I cannot find for certain by whom these Lines were first introduced into Gauging, nor the Manner by which they are laid down; however, what I am going to offer on that Head, seems to be sufficient for the Line S, L, which is by the help of the Table *Pag.* 116. *Edit.* 9. of *Everard's Steriometry:* For the Line of Numbers, or Slider N, (whose Construction has been already shewn) being drawn out till 29.5 thereon (which is the Bung-Diameter of the Cask, from whence this Experiment was made, and on which that Table is built)

I coincide

coincide with a Point at the End of the Line SL, mark'd 100, (which was opposite to 100 on N before it was drawn out) for then against the several dry Inches on the Line N, taken from his Table, must be the Points on the required Line SL, corresponding to the Number of Gallons of the Table that are respectively opposite the dry Inches, by which means thirty one Points of this Line will be had, and the remaining ones may easily be found by Proportion. But if this is not thought sufficient, both it and the Line SS may be made after this manner: Let a Cask be fixed upon, whose Form agrees the nearest to the most common ones in Practice, and suppose both its Bung-Diameter and Length divided into 100 equal Parts, and Planes drawn through each of them, *viz.* the first hundred all parallel to the Heads, by which means this Cask will be divided into a hundred Parts two ways, the first serving for the Line SL, and the second for SS: For if by the preceding Propositions the Measure in Gallons of each of these Slices (which are parallel to the Axis) be successively found, and the last always added to the Sum of the preceding ones, you'll have a Table of Measures analogous to that of Mr. *Everard's*; after which the Line S L may be made, as has been shewn above. In like manner, if the Measure in Gallons in each of the Slices parallel to the Heads of the Casks are successively found, and the last always added to the Sum of the preceding ones, a similar Table will be had for a standing Cask, or for making the Line SS. And this, I imagine, was the Method by which they were actually made, because the Line SL, at this time, does not correspond every where with Mr. *Everard's* Table.

I thought it neceſſary juſt to hint at the Manner by which theſe Lines may be laid down, not ſo much for any Direction to the Inſtrument-Makers, for then I muſt have been more explicit, but that the Officer might be able to give ſome account (if required) of the Invention of every Line on a Rule he ſo frequently uſes. We ſhall now proceed to ſhew the manner of operating thereon.

The firſt Method of ULLAGING.

SECTION I.

To find the Ullage of a ſtanding or lying Cask, by help of the Lines SS *and* SL, *now on the Sliding-Rule.*

The General PROPOSITION.

The Length, Head, and Bung-Diameters, as alſo the wet or dry Inches either in a ſtanding or lying Cask being given, to find the Ullage.

SOLUTION.

Firſt by ſome of the preceding Propoſitions proper to the Caſk propoſed, find its Content; then for a $\left\{\begin{matrix}\textit{Standing}\\ \textit{Lying}\end{matrix}\right\}$ Caſk, ſet the $\left\{\begin{matrix}\textit{Length.}\\ \textit{Bung-Diam.}\end{matrix}\right\}$ on N to 100 on the Line $\left\{\begin{matrix}\mathrm{SS}\\ \mathrm{SL}\end{matrix}\right\}$ and againſt the wet or dry Inches on N, is the *reſerved Number* on $\left\{\begin{matrix}\mathrm{SS}\\ \mathrm{SL}\end{matrix}\right\}$. Again; ſet 100 on B, to the whole Content on A, then againſt the reſerv'd Number on B is the Ullage on A, *viz.* what *remains* in the Caſk, if you uſed the *wet Inches*; but what is *drawn out*, if the *dry Inches*.

Example.

Examp. 1. Let the $\left\{\begin{array}{l}\textit{Length}\\ \textit{Bung-Diam.}\\ \textit{Head-Diam.}\end{array}\right\}$ of a ſtanding Caſk in the Shape of two Fruſtums of a parabolic Spindle be $\left\{\begin{array}{c}40\\ 32\\ 26\end{array}\right\}$ Inches, and the wet Inches 28, to find the Ullage.

The whole Content of this Caſk, by *Prop.* 3. of the laſt *Chap.* is 100.6 Ale Gallons; and 40 the Length on N being ſet to 100 on SS; then againſt 28 the wet Inches, and 12 the dry is 71.6, and 28.4 the reſerved Numbers on SS reſpectively. Again, 100 on B being ſet to 100.6 the whole Content on A, then againſt 71.6 and 28.4, the reſerved Numbers on B, is 72 and 28.6 on A, whoſe Sum is 100.6 the whole Content; the *firſt* is what *remains* in the Caſk, and the *ſecond* what has been *drawn out*.

Examp. 2. Let the Caſk whoſe Dimenſions are the ſame with that above, be lying with its Axis parallel to the Horizon, and here alſo let the dry Inches be 12, and conſequently the wet Inches 20, to find the Ullage.

I ſet 32, the Bung-Diameter, on N to 100 on the Line SL, then againſt 12 and 20 on N is 32.8 and 67.2, the reſerved Numbers on SL. Again, 100 on B ſet to 100.6, the whole Content on A, then againſt 32.8 and 67.2, the reſerved Numbers on B, is 33 and 67.6 on A; the *firſt* is the empty part, or Liquor *drawn* out, and the *ſecond* what *remains*, whoſe Sum is 100.6, the whole Content. Theſe two Examples will be ſufficient to ſhew the Uſe of the Lines SS, SL, now put on the Sliding-Rule for Ullaging; but as they are laid down from ſome particular Caſks, ſo 'tis moſt certain they can only anſwer

for ſuch as are ſimilar to it, or nearly ſo; therefore in the third Method further on, I ſhall ſhew how others may be ſubſtituted in their place, which will render this difficult part of Gauging both readier in Practice, and more certain than heretofore.

The ſecond Method of ULLAGING.

SECTION II.

To find the Ullage *of a Cask by means of a Middle Diameter.*

PROP. I.

The Head-Diameter of an upright Cask, and that of the Liquor's Surface, as alſo another in the Middle betwixt it and the neareſt Head, being given, with the wet or dry Inches, to find the Ullage.

SOLUTION.

To the Square of the Head-Diameter, add that of the Liquor's Surface, and four times the Square of the Middle-Diameter, multiply the Sum by the wet or dry Inches (according to which is leaſt) and that Product for $\left\{\begin{matrix}\textit{Ale}\\\textit{Wine}\end{matrix}\right\}$ by $\left\{\begin{matrix},0004642\\,0005667\end{matrix}\right\}$ or divide it by $\left\{\begin{matrix}2154,32\\1764,71\end{matrix}\right\}$ the Product, or Quotient will be the Ullage ſought, *viz.* what is *drawn out* if the *dry* Inches are *leaſt*, what *remains* in if the *wet* Inches are *leaſt*, and *half* the Content of the Caſk when the *wet* Inches are *equal* to the *dry*.

By the Sliding-Rule thus. For $\left\{\begin{matrix}\textit{Ale}\\\textit{Wine}\end{matrix}\right\}$ ſet $\left\{\begin{matrix}46.41\\42.00\end{matrix}\right\}$ on D, to the Diſtance of the Liquor's Surface, and neareſt Head on C (that is, to the wet or dry

dry Inches on C) then on D find the Diameter of the Head, the Diameter of the Liquor's Surface, and twice the Middle Diameter, noting the three Numbers opposite them on C, whose Sum will be the Measure of the least Frustrum or Ullage required.

Example. Let ABCD (*Fig.* 65.) be a standing Cask, whose Length OO′ is 40 *Inches*, Head-Diameter AD 26, that of the Liquor's Surface MM′ =31,04, the Diameter in the Middle betwixt them, *viz.* NN′=29,06, and OP the dry Inches 12, to find the Ullage.

OPERATION.

The Square of 26 the Head-Diam.	676
The Square of 31,04 the Liquor's Diam.	963,482
The Square of 58,12 (=29.06 x 2) twice NN′= — — —	3377,934
Their Sum — — —	5017,416
Multiplied by the dry Inches —	12
Gives	60208,992
And this multiplied by ,0004642 (*inverted*) — —	2464000,0
	24084
	3612
	241
	12
	27,949

Gives the Ullage or Measure of ADM′M in *Ale Gallons*; which is nearly one Gallon less than was found for this Cask by the Line SS in the last Section, the Dimensions above being such as they would be found in a Cask in the Shape of two equal Frustums of a parabolic *Spindle*. But if you

would try the ſame by the Sliding-Rulet, ſet 46,41 on D, to 12 on C; then againſt 26,31 and 58,12 on D, is 3,76, 5,37, and 18,32 on C, whoſe Sum is 27,95 *Ale Gallons*, as found by Computation. The Dimenſions required in this Method of Ullaging may here eaſily be taken after this manner: The dry Inches OP being taken by dipping, let a ſtreight Ruler with a Plumb-Line at the end, be laid over the Center O, and moved backwards or forwards till the String juſt touch the Caſk as at E, then having made a Mark on the String at S and R, ſo that QS may be equal to the dry Inches, and QR its half; the Parts SM and RN are eaſily meaſured by applying a common Foot-Ruler parallel to QO thro' the Points R,S, and to touch the Staves at N, M; for then twice SM and twice RN taken from twice OQ (which is equal to the Bung-Diameter) will leave MM′ and NN.

Prop. II.

The Length, Bung, and Head-Diameters of a lying Cask, as alſo another in the Middle betwixt them, and the dry or wet Inches being given, to find the Ullage *thereof.*

Solution.

If the Caſk is *more* than half full, from the *dry Inches*, but if *otherwiſe*, from the *wet Inches*, ſubſtract half the Difference of the Bung and Head-Diameters, and alſo half the Difference of the Bung and Middle-Diameter, noting theſe two Remainders, let the leaſt be divided by the Head-Diameter, the biggeſt by the Middle-Diameter, and the wet or dry Inches (according to which is the leaſt) divided by the Bung-Diameter, look for the three Quotients in the Table (of the Areas of the

the Segment (of a Circle) at the End under VS, taking thence the Number opposite them under the Letters *Seg.* multiply the least of these by the Square of the Head-Diameter, the biggest by the Square of the Bung-Diameter, and the remaining one by four times the Square of the Middle-Diameter; and multiply the Sum of these Products by the Length of the Cask, and that for $\left\{\begin{matrix}\textit{Ale}\\ \textit{Wine}\end{matrix}\right\}$ by $\left\{\begin{matrix},000591\\ ,00072\,15\end{matrix}\right\}$ or divided by $\left\{\begin{matrix}1692\\ 1386\end{matrix}\right\}$ the Product, or Quotient will be the Ullage required, *viz.* what is *drawn out* if the wet Inches *exceed* the dry; what *remains in* when the dry inches *exceed* the wet, and *half* the Content of the Cask when they are *equal*.

Example. Let AA′D′D (*Fig.* 66.) be a lying Cask, the Length, Bung, and Head-Diameters 40, 32, and 26, as before; also let NR exactly in the Middle betwixt the Bung and Head-Diameters be 30.5, and the dry Inches, *viz.* EQ=12; to find the Ullage, or Measure of the empty Part EA′*o*′*o*A.

Half the Difference of the Bung and Head-Diameters is 3, half the Difference of the Bung and Middle-Diameters is ,75, both which being respectively taken from 12, the dry Inches, we have 9, 11,25: and 9, 11,25, and 12 being respectively divided by 26: 30,5 and 32 we have for Quotients ,346$\frac{1}{6}$. 368$\frac{5}{6}$ and 375 *nearly*; against which in the Table (when corrected for the fractional Parts) is ,241327—,263052—,269013.

 The

OPERATION.

The first multiplied by 26×26=676 gives - - - - -	163,137
The second multiplied by 61×61= 3721 gives - - -	978,816
The third multiplied by 32×32= 1024 gives - - -	275,469
Their Sum - - -	1417,422
Multiplied by the Length -	40
Gives - - -	56696,8800
And this multiplied by ,000591 (*inverted*) - - -	195000,0
	283484
	51027
	567
Gives the Ullage, or Measure of EA*oo*'A' in *Ale Gallons* -	33,5078

Which is half a Gallon more than was found for this Cask by the Line SL.

Example 2. Let all the Dimensions be the same as above, only here let the Middle Diameter be 29, so that the Cask is conical; here the two Remainders are 9 and 10.5, and the three Quotients 26) 9 (,346$\frac{1}{6}$: 29) 10.5 (.362$\frac{1}{14}$: 32) 12 (,375; against which in the Table (after Correction) is ,241327 ,256539 ,269013.

OPE.

OPERATION.

The first multiplied by 676 gives	163,137
The second multiplied by 3364 gives	862,998
The third multiplied by 1024 gives	275,469
Their Sum is - - -	1301,604
Which multiplied by the Length	40
Gives - - -	52064,1600
And this multipl. by ,000591 (*inv.*)	195000,0
	260321
	46858
	521
Gives the Ullage in *Ale Gallons*	30,7700

This Method is very exact, but too tedious for Practice, unless the Ullage be required to a great Nicety; however, it will serve to compare the former Method, and what I am going to explain in the next place.

The third Method of ULLAGING.

SECTION III.

To find the Ullage *of a Cask by the Mean Diameter.*

PROP. I.

The Length of an upright Cask, the Distance betwixt the Liquor's Surface, and its nearest Head (that is, the wet or dry Inches) and a Diameter taken exactly in the Middle betwixt them, being given, to find the Ullage.

 SOLU-

SOLUTION.

Let the Square of the Middle Diameter be multiplied by the wet or dry Inches, *i. e.* by the least of them, and this Product for $\left\{\begin{matrix}\textit{Ale}\\ \textit{Wine}\end{matrix}\right\}$ by $\left\{\begin{matrix},0027853\\ ,0034\end{matrix}\right\}$ or divided by $\left\{\begin{matrix}359,05\\ 294,12\end{matrix}\right\}$ the Product, or Quotient will be the Ullage required, if the Cask is not half full, or very near it; but when that happens, the mean Diameter of the Cask found by the last Chapter must be used instead of the said Middle-Diameter.

The Reason of this Solution is from hence, that in an upright Cask, not very near *half full*, the lesser Segment may be considered as a Frustum of a parabolic Conoid, (because the bulging Part is supposed to be wanting, which makes this Variety differ so much from the Spheroid and Spindles) and the Measure of every parabolic Conoid is equal to a Cylinder, whose Height is equal to that of the Conoid and Diameter of its Base exactly equal to that Diameter of the Conoid in the Middle betwixt its Ends,

By the Sliding-Rule: For $\left\{\begin{matrix}\textit{Ale}\\ \textit{Wine}\end{matrix}\right\}$ set $\left\{\begin{matrix}18,94\\ 17,14\end{matrix}\right\}$ on D, to the Distance of the Liquor's Surface from the nearest Head on C; then against the middle or mean Diameter on D, is the Ullage on C.

Example. Let the Length of an upright Cask be 40, the dry Inches 12, and a Diameter exactly in the Middle betwixt the Head and Liquid Surface 29 Inches, to find the Ullage or Measure of the empty Part.

OPE-

Operation.

The Square of 29 the Mean Diam. is	841
Multiplied by the dry Inches -	12
Gives	10092,000
And this by ,002785 (*inverted*)	587200,0
	20184
	7064
	807
	50
Gives the Ullage in *Ale Gallons* -	28,105

Or by the Sliding-Rule : Set 18.94 on D, to 12 on C; then againſt 29, the mean Diameter on D, is 28.1 on C, the Ullage required.

I muſt obſerve, this Method for the four firſt Varieties, gives the Ullage a ſmall matter too much, but is exact for the fifth, and a ſmall matter too little for the ſixth; but the Variation from Truth can never be material, but when the Caſk is near half full, and then the mean Diameter of the Caſk is to be taken for the Mean of the Fruſtum, to be meaſured as obſerved before; but theſe laſt Caſes are better performed by the following Propoſition.

Prop. II.

The Length, Bung, and Head-Diameters, as alſo the wet or dry Inches of a ſtanding Caſk, not full, being given, to find the Ullage.

Solution I.

If the Curvature near the Bung be *great*, then multiply the Difference of the Squares of the Bung and

and Head-Diameters, by four times the Square of the Difference betwixt half the Length of the Cask, and the wet or dry Inches; let this Product be divided by three times the Square of the Cask's Length, and the Quotient taken from the Square of the Bung-Diameter; multiply the Remainder by the Difference betwixt the wet or dry Inches, and half the Length of the Cask; then if this Product for $\left\{\begin{matrix}\textit{Ale}\\ \textit{Wine}\end{matrix}\right\}$ be multiplied by $\left\{\begin{matrix},0027 85\\ ,0034\end{matrix}\right\}$ or divided by $\left\{\begin{matrix}359,05\\ 294,12\end{matrix}\right\}$ the Product, or Quotient will shew, how much the Cask is more or less than half full; and therefore, in the first Case, if it be taken from half the Content, you'll have the Quantity drawn out; but in the other Case, added to half the Content, what yet remains in.

2°. *But if the Curvature of the Cask be not great*, multiply the Difference of the Bung and Head-Diameters, by four times the Square of the Difference betwixt the wet or dry Inches, and half the Length of the Cask; let the Product be divided by the Square of the Cask's Length, then this Quotient taken from the Bung-Diameter, will be the Diameter of the Liquor's Surface; which squared and added to twice the Square of the Bung-Diameter, let that Sum be multiplied by the Difference betwixt the wet or dry Inches, and half the Length of the Cask, then this for $\left\{\begin{matrix}\textit{Ale}\\ \textit{Wine}\end{matrix}\right\}$ multiplied by $\left\{\begin{matrix},0009284\\ ,0011333\end{matrix}\right\}$ or divided by $\left\{\begin{matrix}1077,12\\ 882,35\end{matrix}\right\}$ the Product, or Quotient will be, what the Cask is more or less than half full, from whence the Ullage may be found as above.

The

The Reason of both these Solutions is, that that Part of an upright Cask, not full, which lies betwixt the Liquor's Surface and Bung-Circle, may be considered as of the first Variety; and therefore measured by the Rule given for that purpose in the first Chapter, the Diameter of the Liquor in the first Solution being calculated for a Spheroid, and in the second for a parabolic Spindle, as will easily be seen by considering what was said *Corol.* 7, 8. *Chap.* 6. for finding the Middle-Diameter.

We shall illustrate both these Solutions in the *last Example*, where the Length, Bung, and Head-Diameters are 40, 32, and 26, the dry Inches being 12, so that the Difference thereof, from half the Length of the Cask, is 8.

By the first SOLUTION.

The Square of the B. Diam. is 32×32=	1024
The Square of the H. Diam. is 26×26=	676
Their Difference is - - -	348
Multiplied by 4×8×8 - - -	256
	2088
	1740
	696
Gives - - - -	89088
This divided by 3×40×40=4800 gives	18,56
Which taken from the Square of the Bung-Diameter, leaves -	1005,44
This multip. by 8, the Difference betwixt the dry Inches and half the Length, - -	8
Gives - - - - -	8043,520

And

Brought over	8043,520
And this multipl. by ,002785 (*inv.*)	587200,0
	16087
	5630
	643
	40
Gives what the Cask is more than half full in *Ale Gallons* -	22,400
Which taken from half the Content	50,3
Leaves the empty Part = - -	27,9

Nearly the same as found for the same in the preceeding Section.

By the second Method.

The Difference betwixt the Bung and Head-Diameter is - - -	6
Four times the Square of 4x8x8 =	256
Their Product is - - -	1536
Which divided by the Square of the Length - - -	1600
Gives - - - -	,96
This taken from the Bung-Diameter, leaves that of the Liquor's Surface	30,04
Its Square is - - -	963,482
Twice the Square of 32 is -	2048
Their Sum is - -	3011,482
Multiplied by 20—12 = -	8
Gives - - - -	24091,856

This

Brought over	24091,856
This multipl. by ,0009284 (*invert.*)	4829000,0
	~~21683~~
	482
	193
	10
Gives alſo the Exceſs by which the Caſk is more than half full -	22,368
And this taken from half the Content	50,3
Leaves the Liquor drawn out -	27,9320

Nearly the ſame with the true Solution.

So that from hence it appears, if the Caſk is upright and nearly half full, this Propoſition is to be uſed for finding the Ullage, but the preceding one when not near half full.

PROP. III.

The Length, Bung, and Head-Diameter of a lying Caſk, as alſo the wet or dry Inches being given, to find the Ullage.

SOLUTION.

By the laſt Chapter, find the mean Diameter proper to the Caſk propoſed, then to twice the wet or dry Inches, *viz.* the leaſt of the two, or either of them when equal, add the mean Diameter, and from the Sum take the Bung-Diameter; let half of this Remainder be divided by the mean Diameter, and the Quotient looked for under the Letters VS in the Table of the Segment of a Circle at the End, (whoſe Diameter is Unity) from whence take the Number against it under the Letters *Seg.* multiply this by the Square of the mean Diameter, and that Product by the Length of the Caſk, and this for $\left\{\begin{matrix}\textit{Ale}\\ \textit{Wine}\end{matrix}\right\}$ by $\left\{\begin{matrix},003546\\ ,004329\end{matrix}\right\}$ or

or divided by $\left\{\begin{matrix}282\\231\end{matrix}\right\}$ the Product or Quotient will give the Ullage required, *viz.* what is ***drawn out*** if the dry Inches are *least*; what *remains* if the wet Inches are *least*, and half the Content of the Cask when the wet Inches are *equal* to the dry.

Example. Let the Dimension of a lying Cask of the third *Variety* be the same as at the Example of *Prop.* 2. *Sect.* 2. of this *Chap.* and the dry Inches also 12, to find the Measure of the empty Part.

Here the mean Diameter is 30,053 (See *Prop.* 3. *Sect.* 2. of the last *Chap.*) which added to 24, (twice the dry Inches) gives 54,053, and 32 the Bung taken thence, leaves 22,053, half of which is 11,0265; this divided by 30,053, the mean Diameter, gives ,367 *nearly*; against ,367 under VS is

VS is - - - - -	,261284
This multiplied by the Square of 30,053 (*inverted*) - -	81,309
	2351556
	7839
	261
	209
Gives - - - - - -	235,9865
This multiplied by the Length -	40
Gives - - - -	9439,4600
And this multip. by ,003546 (*inv.*)	645300,0
	283184
	47197
	3776
	566
	33,4723

Example.

Example 2. Let the Dimension and the dry Inches be the same as above, only here let the Cask be of the *sixth Variety*.

The mean Diameter here is 29,0522, which added to 24, gives 53,0522; from which 32 (the Bung-Diameter) being taken, leave 21,0522, the half of which 10,5261 being divided by 29,0522, gives $362\frac{1}{3}$ *nearly*; against which in the Table of the Segments is - - - - ,256792

This multiplied by the Square of 29,0522 (*inverted*) - 30,448

	205,4336
	102717
	10272
	77
Gives - - - - - -	216,7402
And this multiplied by the Length	40
Gives - - . -	8669,6080
And this multipl. by 003546 (*inv.*)	6453000,0
	260088
	43348
	3468
	520
Gives the Ullage in *Ale Gallons*	30,7424

From what has been said it appears, the first Example of this Section for a standing Cask, differs but $,\frac{16}{100}$, the second but $\frac{5}{100}$, and the third buut $\frac{16}{1000}$ths of a Gallon from the Truth, as found in the preceding Section; and the two last Examples for a lying Cask are still nearer, for they each of them vary but $\frac{3}{100}$ of a Gallon from what

what was ſhewn for the ſame Dimenſions in the ſecond Method by a Middle Diameter, and is much nearer than can be found by the Lines SS and SL on the Sliding-Rule: But as that Inſtrument is principally uſed in Practice, where the Fatigue of Computation is avoided as much as poſſible, I ſhall ſhew in what follows, how other Lines may be put on the Sliding-Rule, by which the laſt Method for Ullaging a lying Caſk will be rendred very ready in Practice; for it is known by Method 1ſt, now in uſe, there are no leſs than three Operations for finding the Ullage of a lying Caſk to be done on the Sliding-Rule; *Firſt* you muſt find the *whole Content*; *Secondly*, the *reſerved Numbers* by the Lines N and SL; and, *Thirdly*, the *Ullage* by the Lines A, B; whereas, in that I am going to offer, you'll have but two Operations. But I muſt firſt ſhew how theſe Lines are to be made.

The Conſtruction *and* Uſe *of a* New Line *for* Ullaging a *Lying Caſk.*

In order to ſhew the Foundation of this Line, we muſt obſerve, that if S denote any Segment or Number in the Table at the End under the Letters *Seg.* and *d* be the Diameter of any Cylinder, then $d^2 \times S$ is the Area of a ſimilar Segment of its Baſe in Inches. Alſo if L denote the Length of the Cylinder, whoſe Diameter is *d*, then $\frac{d^2 \times S \times L}{282}$ is the Meaſure of a Slice thereof, parallel to the Axis in Ale Gallons, whoſe End is ſimilar to the Segment S; or $\frac{d^2 \times S \times L}{231}$ is the Meaſure of the ſame in *Wine Gallons*; call the Meaſure M: then

then first for Ale Gallons we have $\frac{d^2 \times S \times L}{282} = M$, or $\frac{282}{S} \times M = d^2 L$. Whence this Proportion on the Lines C, D on the Sliding-Rule, *viz.* $\frac{\sqrt{282}}{S}$ on D, is to L on C, so is d on D, to M on C. Whence 'tis apparent, if 282 was divided by every Number of our Table of Segments, made to 1000 versed Sines, (instead of 500) and half the Logarithms of the Quotients laid on a Line such as SL now is, on the Sliding-Rule, from the same Scale of equal Parts that the Line marked N was made from, thence might the Ullage of any lying Cask be found without knowing the whole Content, as now practised; for as the last or least Number on this new Line we are speaking of for Ale Measure is $\frac{\sqrt{282}}{.785398} = 18.95$, so nothing further is requisite after this Line is made, but to bring the mean Diameter of any Cask on N, to 18.95 on this new Line (which would stand where 100 now is,) then against the Mean wet or dry Inches on N, would be a reserved Number on the new Line: And by setting this reserved Number on D, to the Length on C, then against the *mean* Diameter of the Cask on D, would be the Ullage on C, *viz.* what is drawn out, if you used the *mean dry Inches*, but what yet remains in, if you used the *mean wet Inches*: Where, by the *mean wet* or *dry Inches* is to be understood, what remains after half the Difference of the Bung and Mean Diameter is taken from the wet or dry Inches at the Bung. But I must be a little more explicit in this matter. It has already been observed, the first Number towards the right Hand of this Line will

be 18,95, and the Point anſwering to it, the very ſame with that anſwering to 100 now on the Line SL; it is alſo manifeſt the Numbers on our Line will increaſe towards the left Hand, or in a contrary Order to thoſe on the Line N, in ſuch manner, that the Number oppoſite 1, on the Line N, or againſt its Beginning, will be nearly 460,584. The ſame Obſervations ſerve alſo for the Line that is to ſhew Wine Meaſure, only that Number on this at the right Hand will be 17.14, and on the left Hand againſt 1 on N will be a Diviſion nearly anſwering to 416,87; but we may give a Rule for finding a Point on the Line N, againſt which muſt be the Diviſion on the Line we want to conſtruct, agreeing to any given Number.

The Rule is this, To make the Line for $\left\{\begin{matrix}\textit{Ale}\\ \textit{Wine}\end{matrix}\right\}$ Meaſure divide $\left\{\begin{matrix}359,05355\\ 294,11824\end{matrix}\right\}$ by the Square of any given Number; look for the Quotient in Mr. *A. Sharp*'s Table of Segments, and note down the verſed Sine belonging to it; then againſt that verſed Sine on the Line N (not drawn out) is the Point of Diviſion on the new Line, belonging to the ſaid Number, whoſe Square was the Diviſor above.

N. B. I choſe to refer to *Sharp*'s Table of Segments, becauſe it is the largeſt, and moſt correct of any extant. But had this Table been made for a Circle whoſe Diameter (and not its Area) is Unity, then inſtead of the Numbers $\left\{\begin{matrix}359,05 \text{ \&c.}\\ 294,11 \text{ \&c.}\end{matrix}\right\}$ above, we muſt have taken $\left\{\begin{matrix}282\\ 231\end{matrix}\right\}$. I ſhall juſt give an Example of this Method, let the Dimenſions be exactly the ſame with thoſe in the laſt Example.

Ther

Therefore I set 29.05, the mean Diameter on N, to 18.94 on SA; then against 10.52 the mean dry Inches on N, ought to be 33.14 on the new Line AL the *reserved Number*. Again, I set 33.14, the reserved Number on D, to 40, the Length of the Cask on C; then against 29.05, the mean Diameter on D, is 30.7 on C, which is the Ullage required, as was found by Computation.

Hence, from what has been said, 'tis most evident the third Method of Ullaging is by far preferable to what is now practised, whether we regard Truth, or Facility of Operation; so that I do not question but, some time or other, to find it generally practised.

CHAP. III.

Of Gauging Tuns, Backs, Coolers, &c.

THE Vessels called *Tuns* are of various sorts, some having the Ends equal, others unequal but similar, others unequal and dissimilar, but in general their Sides are streight, or such, that a right Line applied from the top to the bottom, touches the same every where, and these are what I shall principally consider in this place: for tho' in some Parts of *England* another sort, where the Sides are not streight, may occur, yet it is very seldom, and when they do, what will be found in the next *Chap. for Gauging Coppers and Stills*, will be sufficient for them also, the Theorems for Measuring all kinds of streight-sided Tuns with paral-

lel Ends is laid down at *Corol.* 5. *Chap.* 6. *Pag.* 192, 193. ſo that all that remains in this Place is but to give the Solutions in Words.

In order to which, we muſt explain the Meaning of ſome Terms uſed herein.

Definition 1. When the Bottom of any Tun is parallel to the Horizon, or exactly level, it is called an *upright Tun*, but otherwiſe, an *inclined one*.

2. What Liquor is required juſt to cover every Part of the Bottom of an inclined Tun, is called the *Drip*.

Or if a Plane be ſuppoſed drawn through the higheſt Point of the Bottom of an inclined Tun, parallel to the Horizon, that part of the Tun contained between the bottom and horizontal Plane, is the *Drip* thereof.

3. That horizontal Plane that juſt touches the higheſt Point of the Bottom of an inclined Tun, is called the *horizontal Baſe*, and another that would juſt touch the loweſt Point of the top or other end, the *horizontal End*.

Thus, (*Fig.* 68.) A*a* being an inclined Tun, where A is the higheſt Point of the Bottom, *a* the loweſt Point of the Top, and AL, *al*, two horizontal Planes drawn thro' A'*a*, then AL is the horizontal Baſe, and *a l* the horizontal End.

4. If from *l* any Point in the horizontal End a Perpendicular be let fall on the horizontal Baſe, *viz.* *l* P, it is called the *Height* of the Tun.

5. The Ends of Tuns are ſaid *to be parallely-poſited*, when the Sides (if Rectangles) or Axis (if Ellipſes) of one End, are parallel to thoſe of the other, *viz.* either when the longeſt Side, or Axis of the one End, is parallel to the longeſt or ſhorteſt Side of the other End.

6,

6. If a parallely-posited Tun is cut thro' the middle of both its Ends by a Plane that is parallel to the Sides when they are Rectangles, or passing thro' the Axis when Ellipses, the Section of this Plane, with the Ends, are called the *Lengths* or *Breadths* of both the Ends respectively; that is, if its Section with one End is called the *Breadth* thereof, then its Section with the other End is called the *Breadth* of that End also; and this tho' it should be longer than the other Dimension of that End.

I was necessitated to introduce these Names for the sake of Brevity, for it is common to call the longest Side of every Rectangle the *Length*, and the shortest the *Breadth*; but then we must have augmented the Number of Propositions in this Section, to prevent Confusion; whereas from hence we can comprehend all the Cases in one.

PROP. I.

The Lengths and Breadths of the Ends of an upright parallely-posited Tun, as also its Height, being given, to find the Measure in Gallons.

SOLUTION.

To twice the Length of the Top, add the Length of the Bottom; multiply the Sum by the Breadth at the Top, and call the Product the *first Number*; also to twice the Length of the Bottom, add the Length of the Top; multiply the Sum by the Breadth at the Bottom, call the Product the *second Number*; add these two Products together, and multiply the Sum by the Height of the Frustum, call that Product the *third Number*; then if the Ends are circular or Ellipses for $\left\{ \begin{array}{l} \textit{Ale} \\ \textit{Wine} \end{array} \right\}$

 multi-

multiply the third Number by $\left\{\begin{matrix},0004642\\,0005667\end{matrix}\right\}$ or divide it by $\left\{\begin{matrix}2154.3\\1764.7\end{matrix}\right\}$ the Product or Quotient will ſhew the Meaſure ſought.

But if the Ends are Rectangles, the third Number multiplied by $\left\{\begin{matrix},0005916\\,0007215\end{matrix}\right\}$ or divided by $\left\{\begin{matrix}1692\\1386\end{matrix}\right\}$ the Product or Quotient will be the Meaſure ſought in Ale or Wine, as above.

This Rule is general for meaſuring any upright parallely-poſited Tun, whether the Ends are Circles, Squares, Ellipſes, Rectangles, Equal or Unequal and ſimilarly poſited (as ſome expreſs themſelves) or not; I ſhall inſtance in two of the moſt difficult Caſes.

Example the Firſt; There is a Tun EF*gh* (*Fig.* 67.) woſe Ends are Rectangles,

The $\left\{\begin{matrix}\textit{Length}\\\textit{Breadth}\end{matrix}\right\}$ of the Top $\left\{\begin{matrix}ab=68\\cd=85\end{matrix}\right\}$

The $\left\{\begin{matrix}\textit{Length}\\\textit{Breadth}\end{matrix}\right\}$ of the Bottom $\left\{\begin{matrix}AB=100\\CD=\ 80\end{matrix}\right\}$

and the Height 30 Inches, whoſe Content is required in Ale Gallons.

OPERATION.

Twice the Length of the Top (68 x 2)	= 136
The Length at the Bottom - -	= 100
Their Sum - - -	= 236
Multiplied by the Breadth at Top	= 85
	1180
	1888
Gives the *firſt Number* - -	= 20060

Twice

Brought over	= 20060
Twice the Length the Bottom -	= 200
The Length at the Bottom -	= 68
Their Sum - - - - -	268
Multiplied by the Breadth at Bottom	= 80
Gives the *second Number* -	= 21440
The Sum of the *1st* and *2d Numb.*	= 41500
Mult. by the Height of the Tun	= 30
Gives the *third Number* - -	1245000
This mult. by ,000591 (*invert.*) -	195000,0
	622500
	112050
	1245
Gives the Meaſ. of the Tun in *Ale Gal.*	= 735.795

In this Example, the longeſt Side of one End was put oppoſite the ſhorteſt Side of the other End, but in that which follows, I ſhall ſuppoſe the longeſt and ſhorteſt Side of one End reſpectively, oppoſite the longeſt and ſhorteſt of the other.

Example 2. Let A *b* (*Fig.* 67.) be an upright parallelly-poſited Tun, whoſe Ends are Ellipſes, its Height being 30 Inches as before; and

The $\left\{\begin{array}{l}\textit{Length}\\ \textit{Breadth}\end{array}\right\}$ of the *Bottom* $\left\{\begin{array}{l}AB = 148\\ CD = 96\end{array}\right\}$ *Inches*;

The $\left\{\begin{array}{l}\textit{Length}\\ \textit{Breadth}\end{array}\right\}$ of the *Top* $\left\{\begin{array}{l}a\,b = 92\\ c\,d = 72\end{array}\right\}$ *Inches*;

Twice the Length of the Top -	= 184
The Length of the Bottom -	= 148
Their Sum - - - -	= 332

 Mul-

Brought over	=	332
Multiplied by the Breadth of the Top	=	72
		664
		2324
Gives the *first Number* - -	=	23904
Twice the Length of the Bottom	=	296
The Length of the Top - -	=	92
Their Sum	=	388
Mult. by the Breadth of the Bottom	-	96
		2328
		3492
Gives the *second Number* -	=	37248
The Sum of the *1st* and *2d* Numb.	=	61152
Multiplied by the Height - -	=	30
Gives the *third Number* -		1834560, ...
This multip. by 000,4642 (*inv.*)	=	2464000,0
		733824
		110074
		7338
		367
Gives the Content in *Ale Gallons*	=	851,603

And from hence the Reader cannot be at a loſs to obtain the Meaſure of any of theſe Tuns when full, or even part empty, by taking the Dimenſions of the Tun at the Liquor's Surface.

But

But as it would be troublesome always to go through with the Computation above, and which indeed, properly speaking, is only fitted for a Tun that is full, it is common in Practice to have the Content of Vessels of this kind for every Inch of their Depth, registered at the Office; so that the Officer has no more to do, but to take the wet or dry Inches; and then from such a Table he has the Liquor in any of them. Hence it is necessary to shew the Method of finding the Content of any Part of these Tuns, usually called *inching* a Tun; which may be performed by the following Proposition.

PROP. II.

The Lengths and Breadths of the Ends of any upright and parallely-posited Tun, together with its Height being given, ti find the Content agreeing to any Number of wet or dry Inches.

SOLUTION.

Multiply the wet Inches by the Length of the Top, and the dry Inches by the Length of the Bottom, the Sum of these Products divided by the Height of the Tun, is the Length of the Liquor's Surface. Also, multiply the wet Inches by the Breadth of the Top, and the dry Inches by the Breadth at Bottom, the Sum of these Products divided by the whole Height of the Tun, is the Breadth of the Liquor's Surface. Thus having got the Length and Breadth of the Liquor's Surface, take those of the Top or Bottom, and by the preceding Proposition find the Content of the upper or lower Frustum; remembring if you use the Lengths and Breadths of the Top with those of the Liquor's Surface, you must also use the dry Inches,

Inches, and the Result will shew what the Tun *wants* of being full; but if you use the Lengths and Breadths of the Bottom with those of the Liquid's Surface, you must also use the wet Inches for the Height, and then the Result will be what Liquor remains in the Tun.

Corol. 1. Hence by supposing 1, 2, 3, 4, &c. for the dry Inches, and these substracted from the whole Height the wet Inches, you may by this Rule inch any streight-sided Tun, whose Ends are parallely-posited, be they round, square, elliptical, or rectangular, similar or dissimilar.

Corol. 2. But from the Forms of these Tuns, it is evident the several Lengths are equidifferent, and so are the Breadths; so that one of each from the Top or Bottom being found, you'll from thence have their common Difference, which continually added to, or substracted from the preceding one, as the Case requires, will give the succeeding ones.

Examp. Let there be an upright parallely-posited Tun, whose Ends are Ellipses, the Length at $\left\{\begin{matrix} Top = 65 \\ Bott. = 110 \end{matrix}\right\}$ Inch. the Breadth at $\left\{\begin{matrix} Top = 60 \\ Bott. = 100 \end{matrix}\right\}$ Inches, and the whole Depth 12 Inches, to find the Length and Breadth at 1 Inch from the Top.

OPERATION.

The wet Inches multiplied by the Length at Top, *viz.* -	$11 \times 65 = 715$
The dry Inches multiplied by the Length at Bottom, *viz.* -	$1 \times 110 = 110$
Their Sum is - - - -	$= 825$
Which divided by 12, the whole Height, gives - -	$68\frac{3}{4}$

There-

Therefore $68\frac{3}{4}-65=3\frac{3}{4}$ the common Difference of all the Lengths.

Alſo the wet Inches mult. by the Breadth at Top, *viz.*	$11\times60 = 660$
The dry Inches multiplied by the Breadth at Bottom, *viz.*	$1\times100 = 100$
Their Sum is	$= 760$
Divided by 12, the whole Depth, gives	$=63\frac{1}{3}$

Therefore $63\frac{1}{3}-60=3\frac{1}{3}$ the common Difference of all the Breadths.

Or, If the Difference of the Lengths at Top and Bottom, *viz.* $110-65=45$, had been divided by 12, the whole Depth, the Quotient ($3\frac{3}{4}$) is the Difference of all the Lengths; and the Difference of the Breadth at Top and Bottom, *viz.* $100-60=40$ divided by 12, gives $3\frac{1}{3}$ for the Difference of all the Breadths, as found before. Therefore $3\frac{3}{4}$ added to 65, and to the Sum continually, gives all the Lengths, as in the 2*d* Column of the following Table; and $3\frac{1}{3}$ added to 60, and to the Sum continually, gives all the Breadths as in the 3*d* Column.

Corol. 3. We may alſo give a Rule for computing the Contents belonging to the ſeveral wet or dry Inches, from having three contiguous Contents given, which renders the inching of theſe ſtreight-ſided Tuns exceeding eaſy.

Rule. Let each of the three contiguous Contents be divided by its dry Inches, and the ſecond Quotient taken from the third; then if three times the Remainder be added to the firſt Quotient, and the Sum multiplied by the dry Inches belonging to the Content required, the Product will be that Content.

We will illuſtrate this in inching the Tun mention'd in the laſt Example; and in the firſt place

I we

we muſt find (by *Prop.* 1.) the Contents belonging to the three firſt dry Inches, the Operation for them is thus, *viz.* for the

$$\textit{1ſt.}\left\{\begin{array}{l}\overline{\overline{65\times2+68\frac{1}{4}}\times60+\overline{68\frac{1}{4}\times2+65}\times63\frac{1}{3}}\times\\ \dfrac{1}{6\times359{,}05}=11.4884 \textit{ Ale Gallons.}\end{array}\right.$$

$$\textit{2d.}\left\{\begin{array}{l}\overline{\overline{65\times2+72\frac{1}{2}}\times60+\overline{72\frac{1}{2}\times2+65}\times66\frac{2}{3}}\times\\ \dfrac{2}{6\times359{,}05}=24.2766 \textit{ Ale Gall.}\end{array}\right.$$

$$\textit{3d.}\left\{\begin{array}{l}\overline{\overline{65\times2+76\frac{3}{4}}\times60+\overline{76\frac{3}{4}\times2+65}\times70}\times\\ \dfrac{3}{6\times359{.}05}=38.4340 \textit{ Ale Gall.}\end{array}\right.$$

And theſe three Terms being found, will ſerve to find all the reſt without purſuing the above Operations any further; for call the firſt A, the ſecond B, the third C, and the reſt in order D, E, F, G, *&c.* then by the laſt Rule we have

$$\left.\begin{array}{l}D=\overline{\frac{A}{1}+3\times\overline{\frac{C}{3}-\frac{B}{2}}}\times4=54{,}0305\\ E=\overline{\frac{B}{2}+3\times\overline{\frac{D}{4}-\frac{C}{3}}}\times5=71{,}1355\\ F=\overline{\frac{C}{3}+3\times\overline{\frac{E}{5}-\frac{D}{4}}}\times6=89{,}8187\end{array}\right\}\textit{The Content at}\left\{\begin{array}{c}4\\5\\6\end{array}\right\}\textit{Inches from the Top.}$$

And after this manner the Numbers in the 4*th* Column of the following Table were computed. Each of which taken from 238,9595, the whole Content of the Tun leaves thoſe in the 5th Column.

The

The TABLATURE *for Inching an upright and paralelly-posited Tun, whose Ends are Ellipses.*

Dry Inch.	Lengths	Breadth	Cont. from Top	Cont. from Bottom	Wet Inch.
0	65	60	0	238,9595	12
1	$68\frac{3}{4}$	$63\frac{1}{3}$	11,4884	227,4711	11
2	$72\frac{1}{2}$	$66\frac{2}{3}$	24,2766	214,6829	10
3	$76\frac{1}{4}$	70	38,4340	200,5255	9
4	80	$73\frac{1}{3}$	54,0305	184,9290	8
5	$83\frac{3}{4}$	$76\frac{2}{3}$	71,1355	167,8240	7
6	$87\frac{1}{2}$	80	89,8187	149,1408	6
7	$91\frac{1}{4}$	$83\frac{1}{3}$	110,1497	128,8098	5
8	95	$86\frac{2}{3}$	132,1982	106,7613	4
9	$98\frac{3}{4}$	90	156,0338	82,9257	3
10	$102\frac{1}{2}$	$93\frac{1}{3}$	181,7262	57,2333	2
11	$106\frac{1}{4}$	$96\frac{2}{3}$	209,3446	29,6147	1
12	110	100	238,9595	0	0

The Use of this Table is evident by Inspection; for suppose we should come to the above Tun, and find 7 wet, or 5 dry Inches, then to find what Liquor is taken out or yet remains in the same, 'tis but looking for 7 in the last Column, or 5 in the first, and opposite either in the 4th is 71,1355, what the Tun wants of being full, and in the 5th Column is 167,8240 what Liquor is in the same.

Corol. 4. But since Tuns generally are made to lean a little, and also have a certain Point fixt for taking the wet or dry Inches from, call'd the *Dipping-Place*, which is commonly so taken, as to be the easiest come at; we must shew how these Tuns are to be inched, and a Table made thereof; in order to which, and for the sake of Brevity,

we

we ſhall refer to *Fig.* 68. where AL *a l* are the horizontal Ends of the inclin'd Tun A *a*, and their Diſtance, or the Height thereof, *l* P; then by the preceeding Method, the Part AL *a l* being inch'd and a Table made of the ſame, as has been ſhewn already, let the Meaſure of the *Hoof* or *Drip* ALB be found, by the general *Prop. pag.* 172. and its Corollaries; or otherwiſe, alſo from the aſſign'd Dipping-place, take the wet Inches belonging the Drip, then if this is added to all the wet Inches of AL *l a*, reckoning from the horizontal Baſe AL, you'll have the wet Inches for the inclin'd Tun, alſo the Meaſure of the Drip added to each of the Meaſures in the 5th Column will be the Contents of thoſe wet Inches. Then theſe ſeveral Contents taken from the whole Content of the Tun, contain'd betwixt the real Baſe AB, and Horizontal End *a l*, will leave the Contents of the dry Part, wanting the Hoof at Top; and the wet Inches taken from the whole Dip, from the Dipping-place to the Bottom of the Tun, will leave the Dry Inches correſponding.

To illuſtrate what has been ſaid, let *Fig.* 58. be a Tun whoſe horizontal Ends AL, *a l*, are the ſame with that we have inch'd above; its Height being *l* P=12 Inches; alſo let the Dipping-place be at D, and the whole Dip D*p*=15 Inches; of which ſuppoſe that Part betwixt *p* and the horizontal Bottom is 2 Inches, and the Meaſure of the Drip 15 Gallons; then by the above Method we have the following Table.

Dry

Dry Inch.	Lengths	Breadths	Content from *a l*	Content from AB	Wet Inch.
1	65	60	0	253,9595	14
2	$68\frac{3}{4}$	$63\frac{1}{3}$	11,4884	242,4711	13
3	$72\frac{1}{2}$	$66\frac{2}{3}$	24,2766	229,6829	12
4	$76\frac{1}{4}$	70	38,4340	215,5255	11
5	80	$73\frac{1}{3}$	54,0305	199,9290	10
6	$83\frac{3}{4}$	$76\frac{2}{3}$	71,1355	182,8240	9
7	$87\frac{1}{2}$	80	89,8187	164,1408	8
8	$91\frac{1}{4}$	$83\frac{1}{3}$	110,1497	143,8098	7
9	95	$86\frac{2}{3}$	132,1982	121,7613	6
10	$98\frac{3}{4}$	90	156,0338	97,9257	5
11	$102\frac{1}{2}$	$93\frac{1}{3}$	181,7262	72,2333	4
12	$106\frac{1}{4}$	$96\frac{2}{3}$	209,3448	44,6147	3
13	110	100	238,9595	15	2

This Method ſeems to me more natural than that now practiſed, where ſomething is generally to be added or ſubſtracted, as is expreſſed in the Dipping-place: for here by entring the Table with any dry or wet Inches (taken at the Dipping-place) in the firſt or laſt Column, then in the *4th* and *5th* you have what is wanting to fill the Tun as it then ſtands, and what is then in the ſame, and this directly without being troubled to add or ſubſtract at all.

SCHOLIUM I.

The Quantity of the Drip is commonly found in meaſuring as much Water, as will juſt cover the Bottom, and then alſo an Account may be taken of the Heights to which the Liquor riſes, in pouring in every Gallon, ſo as to be able to tell how much is in when the Bottom is not cover'd; for in all

all other cases if the Depth or Dip is not an exact Number of Inches, you may find how much is to be added or substracted from the tabular Number by Proportion. Thus in the last Example, suppose we should find at any time the wet Inches 7, 2, take the Content for 7 wet Inches from that for 8, and there remains 15,6050; then say if 1 gives 15,6050, ,2 will give 3,12100, and so much is to be added to 167,8240, which gives 170,945, *nearly* the Content belonging 7,2 wet Inches.

We have observed, if the Bottom of the Tun is even, and also the Tun it self the Frustum of any Pyramid or Cone, the Quantity of the Drip may be computed by the Theorem, *Page* 174. and its Corollaries: but since the Inclination is commonly but small, half the Measure of the Frustum ABML, (where ML is parallel to AB′) may be taken for the Measure of the Drip; for tho' that will always be a little different from Truth, unless in prismatic or cylindric Tuns, or Coolers, yet the Error will not be of any Consequence in Practice.

SCHOLIUM II.

Backs, or Coolers, whose Sides are commonly at Right Angles to their Bottoms, which are always very broad in respect of their Height, that so the Liquor put into them, may the sooner cool; I say these Vessels are not only inch'd, but the Content found to every tenth of an Inch, which is called *tenthing* a Back; in order to which, the exact Depth of the Vessel must be found, for the Bottoms being very broad, are very often uneven, and thence the Depths taken at different Places must be different, a small Error in which would produce a considerable one in the Content: Therefore to find the true Depth, or Dip

Dip of a Back, or Cooler, let any Number of Dips be taken, and all added together, then the *Sum* divided by the *Number* of Dips, will be the true Dip; after which you muſt try round the Veſſel, till you find a Place where the Dip is equal to that you before found for the mean Dip, which muſt be mark'd for the Dipping-place. But this may be aſſumed any where, for 'tis but marking how much the Dip taken there, differs from the mean Dip, and noting the Difference thus, let the mean Dip of any Back be 5 Inches, and by dipping at the aſſumed place, ſuppoſe we ſhould find 5,5 for the Dip there, thence 'tis evident, 5 muſt be taken from all the wet Dips, and it may be noted down thus D—,5; but had the Dip there been found 4,5, which is 5 leſs than the mean Dip, then the Dipping-place muſt be mark'd D+,5.

The Dipping-place and mean Dip being found, and mark'd as above, we ſhall next ſhew how the ſame is to be tenth'd; in the firſt place, let the Area of the Top or Bottom be found by ſome of the preceding Propoſitions, then this divided by 282, will ſhew how many Gallons the Back holds at 1 Inch depth, one tenth of which will be what it holds at $\frac{1}{10}$th of an Inch depth; this doubled, tripled, quadrupled, &c. will be the Contents at $\frac{1}{10}$th, $\frac{2}{10}$ths, $\frac{3}{10}$ths, $\frac{4}{10}$ths, &c. of an Inch of its Depth; and by repeating this Operation, 'till you have multiplied the Content at $\frac{1}{10}$th of an Inch in depth by the mean Dip, you'll obtain a Table of the Contents of the whole Back for every tenth of an Inch of its Depth, the laſt of which will be the whole Content of the Veſſel.

For Example, Let there be a Back reſembling a Parallelopipedon, whoſe Length is 140 Inches, Breadth 120 Inches, and let the Depths taken at

5,1
4,8
5
5,2
4,6
4,9
4,5
4,7
5,2
4,9
——
48,9

ten different Places cross-ways, be as in the Margin, where the Sum is 48,9, this divided by 10, the Number of Dips, or which is here the same thing, removing the decimal Point one place towards the left Hand, gives 4,89 for the mean Depth.

Next to tenth the same, because the Base of this Back is a Rectangle 140 x 120= 16800 is the Area thereof in Inches, which divided by 282, gives 59,57451, for the Content in Gallons; at 1 Inch depth therefore 5,95745 is the Content at $\frac{1}{10}$th of an Inch depth; and this reduced into Barrels, Firkins and Gallons, is 0, 0, 5,95745; 8 Gallons being 1 Firkin, and 4 Firkins 1 Barrel.

B. F. A. Gall.

This set down in the following Manner, *viz.*

Depth	*B.*	*F.*	*Content in Ale Gal.*
,1 -	0 :	0 :	5,95745
,2 -	0 :	1 :	3,91490
,3 -	0 :	2 :	1,87235
,4 -	0 :	2 :	7,82980
,5 -	0 :	3 :	5,78725
,6 -	1 :	0 :	3,74470
,7 -	1 :	1 :	1,70215
,8 -	1 :	1 :	7,65960
,9 -	1 :	2 :	5,61705
1,0 -	1 :	3 :	3,57450
4,89 -	9 :	0 :	3,31936

And multiplied by 2, 3, 4, 5, &c. to 10, gives the Contents of this Back for every tenth of an Inch in Depth, which is sufficient for Use, so we have only to the tenthing for an Inch, added the

the whole Content, which is found by multiplying the last Number by the mean Depth 4,89.

And this will be sufficient for Backs whose Ends are equal similar Figures, and their Sides streight; but should they be otherwise, 'tis necessary to find the Content for every Inch of the Depth, by taking the Dimensions in the Middle of every one of them, after which you may proceed as above, only the Operation, or the Number that is to be multiplied by 2, 3, 4, 5, 6, &c. will be different for every Inch of Depth, but the *Manner* is the same in all.

Of Gauging MALT.

In making Malt they use a Cistern, Couch and Floor, each of which is generally in the Shape of a Parallelopipedon, so that if the Area of the Base of any of them in Inches be multiplied by the Depth of the Malt, (in the same Measure,) you will have the Quantity of Malt in cubic Inches, and this divided by 2150,42, or multiplied by ,0004651, the Quotient, or Product will be the Measure of the same in Bushels. But we must observe that Malt being a dry Substance, does not like what is Liquid, form a Plane at the Top, parallel to the Horizon, so that 'tis the Officer's business to examine carefully if the Malster has laid his Malt even; if not, he must consider what is to be allow'd for the Irregularity: Now the Method to do that, is the same with what we gave before for taking the mean Depth of a Cooler, which is, by taking a sufficient Number of Depths crossways, and at each Angle of the Figure form'd thereby, for then the *Sum* of all the Depths divided by their *Number*, will give the *mean Depth*,

or ſuch a one, that the Area of the Baſe being multiplied thereby, and that by ,0004651, this laſt Product will give the Quantity of Malt in Buſhels.

How this may be effected by the Sliding-Rule, has been ſufficiently explained before, (*Pag.* 52, 53, 54, 55.) ſo that it would be needleſs to inſiſt any longer on it in this Place.

CHAP. IV.

Of Gauging Coppers, Stills, *and* Curve-ſided Tuns.

I. *To Gauge a Copper which is Broadeſt at Top.* See Fig. 69.

Rule. TAKE Croſs-Diameters, that is, two perpendicular to each other, at the Top (AB) at the Bottom (*q q*, where a Plane would juſt touch H, the Vertex of the Crown) and alſo others (at K) exactly in the middle betwixt the Top (AB) and Bottom (*q q*). Then add the Product of the Top-Diameters, the Product of the Bottom-Diameters, and four times the Product of the Middle-Diameters together, let this Sum be multiplied by (GH) *viz.* the Height of the Copper from its Top (G,) to the Vertex of the Crown (H,) and that by ,0004642, or divide it by 2114,32, the Product, or Quotient will be the Meaſure of the Copper, all but what is required to cover the Crown; and this muſt be found by meaſuring in Water: then this added to the former, will ſhew the Quantity of Liquor the ſaid Copper will hold.

II.

II. *To measure a Still, or Copper, when the Top is not the Broadest.* (Fig. 70.)

Rule. Suppose the same divided into two Frustums, by a Plane passing thro' the broadest Part (LL,) and let each of these Frustums be measured by the last Rule, then the Liquor requir'd to cover the Crown being added to the Sum of the Measures of the two Frustums will be the Measure of the Copper or Still proposed.

This Solution is deduced from *Corol.* 4. to *Prop.* 6. *pag.* 191. But if the Vessel is very large, you may divide the Distance betwixt the Top and most bulging Part, as also that betwixt the Bottom and most bulging Part into 2, 3, or more equal Parts, and take Cross-Diameters at each Point, then find the Measure by using the Theorem standing against 4 or 5 Ordinates, *Page* 187. but the Rules above will be found generally sufficient for gauging any Still, Copper, or Curve-sided Tun to be met with in Practice, therefore we shall proceed to illustrate them by Examples.

Examp. I. Let ABCD (*Fig.* 69.) be a Copper broadest at the Top, whose Height from the Top G, to the Vertex of the Crown H is 36 Inches, and let the Point K be taken exactly in the Middle betwixt G and H; and suppose the Cross-Diameters at $\left\{\begin{matrix}H\\K\\G\end{matrix}\right\}$ be $\left\{\begin{matrix}110,0\\113,0\\116,0\end{matrix}\right\}$ and $\left\{\begin{matrix}111,0\\114,0\\115,5\end{matrix}\right\}$ then for this Copper we have the following Operation.

The Rectangle of the Top-Diameters, *viz.* 116 x 115,5 =	13398,0
The Rectangle of the Bottom-Diamet. *viz.* 110 x 111 =	12210,0
4 Times the Rectang. of the Mid. Diam. *viz.* 113 x 11 4 x 4 =	51528,0
Their Sum - - - -	77136,0

 Mul-

Brought over	77136,0
Multiplied by the Height - -	36
	462816
	231408
Gives - - - -	2776896
And this by ,0004642 (*inv.*) -	2464000,0
	11187584
	1666138
	111076
	5554
Gives the Meaſure of the Part ABqq in *Ale Gallons* -	1289,0352
To which add the Liquor requir'd to cover the Crown -	28,
And you'll have the Content of the whole Copper, *viz.* -	1317,0352

Examp. II. Let ABCD (*Fig.* 70.) be a Tun where the greateſt Bulge is not at the Top AB, but lower down as at LL, and let the whole Height GH be 36 Inches, as before; of which the Diſtance of the Bulge LL, *viz.* GO is 15 Inches, and conſequently OH=21 Inches, alſo let the Points KK′ be taken in the middle betwixt the Points GO and OH, and Croſs-Diameter taken at the Points G, K, O, K′, H, which ſuppoſe as here under; *viz.* let the Croſs-Diameters

at	G	be	80,5	and	80,8	then for this Tun
	K		85,5		86,	
	O		89		89,5	
	K′		86,5		87,	
	H		83		83,5	

we have the following

Ope-

OPERATION.

The Rectangle of the Top-Diameters, *viz.* 80,5 x 808 =	6504,4
The Rectan. of the greateſt Bulge-Diameters, *viz.* 89 x 89,5 =	7965,5
4 Times the Rectang. of the Mid. Diam. at K, *viz.* 85,5x86x4=	29412,0
Their Sum - - - - -	43881,9
Multiplied by the Height -	15
	2194095
	438819
	658228,5
And this by ,0004642 - -	2464000,0
	26329
	3949
	263
	13
Gives the Meaſure of Part ABLL in Ale Gallons - -	305,54

Again,

The Rectangle of the Bottom-Diameters, *viz.* 83x83,5 = -	6930,5
The Rectan. of the greateſt Bulge-Diameters, *viz.* 89 x 89,5 =	7965,5
4 Times the Rectang. of the Mid. Diam. at K', *viz.* 86,5 x 87, x 4=	30102,0
Their Sum - - - -	44998

	Brought over 44998
Multiplied by the Height - -	21
	44998
	89996
	944958,00
And this by ,0004642 (*inverted*)	2464000,0
	37798 3
	5669 7
	378 0
	18 9
Gives the Measure of Part CDLL in Ale Gallons - -	438,649
To which add the Measure of the Part ABLL - - -	305,54
Then add the Liquor required to cover the Crown - -	32
And you'll have the Content of the whole Tun in Ale-Gallons -	776,189

Examp. 3. Let ABCD (*Fig.* 71.) be a Still where the greatest Bulge is at LL, and let the whole Height GH be 42,8 Inches, of which the Distance of the Bulge LL, *viz.* GO is 22,3 Inches, and consequently OH=20,5 Inches; also let the Points K, K', be taken in the Middle betwixt the Points GO, and OH, and Cross-Diameters taken at the Points G, K, O, K', H, which suppose as below, *viz.* let the Cross Diameters

$$\text{at}\left\{\begin{matrix}G\\K\\O\\K'\\H\end{matrix}\right\}\text{be}\left\{\begin{matrix}21\\45,4\\47,8\\47\\44\end{matrix}\right\}\text{and}\left\{\begin{matrix}21\\46\\47,3\\47\\44\end{matrix}\right\}\text{then for this Still}$$

we have the following Ope-

OPERATION.

The Rectangle of the Top Diam. viz. 21 x 21 = - - -	441
The Rectan. of the Diameters of the greatest Bulge, viz. 47,8 x 47,3=	2260,94
4 Times the Rectangle of the Mid. Diams. at K, viz. 45,4 x 46 x 4 =	8353,6
Their Sum - - - - -	11055,54
Multiplied by the Height -	22,3
	3316662
	2211108
	2211108
Gives the Measure of Part ABLL *	246538,542

Again,

The Rectangle of the Bottom Diameters, viz. 43,5 x 44 =	1914
The Rectan. of the Diameters of the greatest Bulge, viz. 47,8 x 47, 3 =	2260,94
4 Times the Rectangle of the Mid. Diameters, viz. 47 x 47 x 4 =	8836,
Their Sum - - - -	13010,94
Multiplied by the Height - -	20,5
	6505470
	2602188
-	266724,270
*	246538,542
	513262,812

This

Brought over	513262,812
This multiplied by 0005666 (*inv.*)	6665000,0
	256631
	30796
	3079
	308
	290,814
Then add the Liquor required to cover the Crown - -	7,5
And you'll have the Content of the whole Still in Wine Gallons -	298,314

The Rule made use of (in or about *London*) for gauging Stills, is to measure the globical Frustum, or Part, from the Nails to the Top, as by *Rule* 3. at *Page* 132. foregoing; and then the Body, or remaining Part, by taking Cross-Diameters at the Middle of every 6, 8, or 10 Inches: from thence their corresponding Areas are found, and multiply'd into their respective Depths, will give as many Cylinders, as are the Numbers of Diameters taken; the Sum of which Cylinders, added to the globical Frustum above, and the Liquor required to cover the Crown, will give the Content of the Still required.

As for Example, Let ABCD (see *Fig.* 71.) be a Still, whose Depth GH is 42,8, and GK=7,3, consequently KH will be 35,5, and their respective middle Cross-Diameters as follows.

OPE-

OPERATION.

Inches	Diameters	Diameters	Areas		Wine Gall.
7,3	21,0	21,0	1,50	$5,75 + 5,75 + 1,50 \times \frac{7,3}{3}$	= 31,59
	41,5	40,8	5,75		
6,0	43,9	43,2	6,45 × 6		38,70
6,0	47,0	46,2	7,38 × 6		44,28
6,0	47,8	47,3	7,69 × 6		46,14
6,0	47,6	47,4	7,67 × 6		46,02
6,0	46,5	46,5	7,35 × 6		44,11
5,5	45	45,2	6,91 × 5,5		38,03
Depth 42,8				To cover the Crown	7,50
				Content	296,37

This Rule for the globular Part holds true in no case, except when the lower Diameter is the Diameter of the generating Circle, which was justly remarked to me some time ago, by my ingenious Friend Mr. *Geo. Powel*, who has had many Years

Years Experience in surveying the Distillery, and is well skill'd in these kind of Speculations, and has clearly demonstrated, if four times the Square of the Altitude be not equal to the Difference of the Squares of the Diameters, (which he says he never yet found) the greatest Diameter will not be that of the Sphere.

I shall here subjoin his Demonstration, for the Satisfaction of those chiefly, who are concern'd in that Branch of the Revenue, and have so much Curiosity as to know the true Reason of what so frequently occurs, and would not rely too much upon the Force of Custom, and take things for granted upon an implicit Belief only, when there is to be had most evident and indubitable Principles from Geometry, to found a certain Theory upon. *

SCHOLIUM I.

To inch down curve-sided Figures, as Stills, &c. representing the Frustum of a Sphere, observe the following Articles.

I. As eight times the Height of the proposed Frustum, in Inches, is to the Sum of its Base and Top-Diameters, so is the Difference of those Diameters to a fourth Number; from which Number increased by one Inch, take half the said Height, and multiply the Remainder by ,0272, reserving the Product.

II. Multiply the Square of half the Base, or greater Diameter, by ,0136, and from the Product increased by ,0090666, *&c.* take half the reserved

* *Fig.* 7. *Plate* 2. Let AB = greater Diameter, EK the lesser, and DC = Height; then $\frac{AB^2}{4} - \frac{EK^2}{4} = DC^2$, that is $AB^2 - EK^2 = 4DC^2$. *Q. E. O.*

reserved Product, the Remainder will be the Wine Gallons at 1 Inch deep, which also reserve.

III. To the reserved Product found by Article I. add ,0272, and to that Sum add also ,0272, and to this last Sum ,0272, and so on continually, till the Number of Additions, or Sums thus found, are equal to the Number of Inches in the Height of the Frustum lessen'd by two.

IV. From the reserved Remainder take the reserved Product, and from the Remainder thus arising, take the first of the said Sums (mention'd in Article III.) and from the next Remainder the second of those Sums, and from the next the third, and so on continually.

V. Add the Remainders, mentioned in the last Article, continually together, in order as they follow; the Sum of the two first will be the Wine Gallons contained at 2 Inches deep, of the three first at three Inches deep, *&c.*

Example. Let the Top-Diameter be 21, and the Base 41, and the Height of the Frustum 7 Inches.

Then it will be as 56 : 62 : : 20 : 22142, from which increased by 1, taking 3,5, there rests 19,642; and this multiplied by ,0272, gives ,53421 for the reserved Product: And therefore the Square of half the greater Diameter by ,0136 being 5,715400, we have 5,4573354, equal to the reserved Remainder.

The Work will stand as follows;

Sums

Sums.	Remainders	Inch.	
,53426	5,4573	1	5,45733
,0272	,5343		4,92307
,56146	4,9230	2	10,38040
,0272	,5614		4,36161
,58866	4,3616	3	14,74201
,0272	,5886		3,77294
,61586	3,7730	4	18,51495
,0272	,6158		3,15708
,64306	3,1572	5	21,67203
,0272	,6430		2,51402
,67026	2,5142	6	24,18605
	,6702		1,84376
	1,8440	* 7	26,0281

SCHOLIUM II.

To take the Dimensions of a Still.

First take its true Depth from the Center of the rising Crown H, to the Center of the Collar g, by laying a flat Rule, or the like, across the Middle thereof, which being done, you must deduct the Depth of the Collar from the Depth so taken; by which means you'll obtain the true Height GH. At the same time you may also find the Altitude of the globical Part or (Zone) ABEF, which is join'd to the Body of the Still EFCD with large Nails, so that the Place of Contact plainly appears by a Seam that goes quite round, and this is usually done by extending a string diametrically in the Seam EF; and by observing where

where it cuts your Dimension-Rod in K; if you take that Number from the whole Depth, it is evident you'll not only have the Height of the Zone KG, but also KH. In the next place, quarter the Still, and take Cross-Diameters *dd*, *rr* in the Middle of every six, eight, or ten Inches, the latter is generally sufficient and most expeditious.

If you make use of *Theorem Page* 191. at the greatest Bulge, or broadest Part of the Still LL. At the Top AB, and the Bottom *qq*, where a Plane would just touch H, the Vertex of the Crown (by the help of your Dimension-Rod) take Cross Diameters, also others, exactly in the Middle betwixt the greatest Bulge LL, and each End, *viz.* AB and *qq*, as at KK′, which are all the Dimensions required; after this manner the same may be obtain'd for Coppers and curve-sided Vessels.

ADDENDA.

I. Tho' we have already given all the Rules for gauging Tuns and Casks commonly practised, with some others that have not been publish'd before; yet I thought it would not be improper to add what follows, which in some Cases may be of use.

The Theorems I am going to offer, are for gauging Casks whose Heads are unequal, and consequently unequally distant from the Bung or bulging Part, in which Class some or most Stills, Coppers, and Tuns with curved Sides may be rank'd.

Let ABCD (*Fig.* 72.) be a Vessel made from some of the Varieties mention'd *Page* 198. by cutting off a Part O′P from one End, and put $\left\{\begin{array}{l} AD=b \\ BC=H \\ EF=b \\ OO'=l \end{array}\right\}$

Then if the original Cask was of the

$$\begin{array}{l} 1st \\ 3d \\ 5th \\ 6th \end{array}\Bigg\}\ \text{Variety of K} = \left\{\begin{array}{c} \frac{l\sqrt{b^2-H^2}}{\sqrt{b^2-H^2}+\sqrt{b^2-b^2}} \\ \frac{l\sqrt{b-H}}{\sqrt{b-H}+\sqrt{b-b}} \\ \frac{l\times\overline{b^2-H^2}}{2b^2-H^2-b^2} \\ \frac{l\times\overline{b-H}}{2b-H-b} \end{array}\right\} = \text{KO} = \left\{\begin{array}{c} \frac{l\sqrt{b^2-b^2}}{\sqrt{b^2-H^2}+\sqrt{b^2-b^2}} \\ \frac{l\sqrt{b-H}}{\sqrt{b-H}+\sqrt{b-b}} \\ \frac{l\times\overline{b^2-b^2}}{2b^2-H^2-b^2} \\ \frac{l\times\overline{b-b}}{2b-H-b} \end{array}\right.$$

And having by these Theorems got the Distance of each Head from the bulging Part, the Content of both these Frustums will be found by *Chap.* 1. *Part.* 3. whose Sum will be the Measure of the Vessel proposed.

Corol. From these Theorems 'twill be easy to deduce Rules for computing a Diameter at a given Distance from the Bung-Circle for any of those Varieties, and from thence the Reason of the Rule for Ullaging a standing Cask, *Page* 233.

3. *To find the true Content of the Globical Part of a Still in Wine Gallons.*

* *Rule.* To half the Sum of the Squares of the Top and Bottom Diams. add $\frac{2}{3}$ of the Square of the Altitude, multiply the Sum by the Depth, and this Product divided by 294, or multiplied by ,0034, and you'll have the Content in Wine Gallons.

Examp. Let the Top-Diameter be 21, Bottom-Diam. 41,2, and the Height 7,3 Inches.

OPERATION.

$21\times21=441$, $41{,}2\times41{,}2=1697{,}44$, and $7{,}3\times7{,}3\times\frac{2}{3}$

$\therefore =35{,}52$ $\overline{\frac{441+1697{,}44}{2}+35{,}52}\times\frac{7{,}3}{294} = 27.48$ Wine Gallons, *nearly.*

II. *The Method for computing Logarithms.*

We shall now give the Rules for computing the Tabular Logarithms, as promised *Page* 31. In order to which, let a, b be any two Numbers, of which b is the greatest, put $a+b=s$, and $b-a=d$, also assume $n=,8685889638$ &c. then the Tabular Log. of a, b are as follows, *viz.*

L : *a*

$$L:a = L:b - \frac{n \times d}{s} - \frac{d^2}{3s^2}A - \frac{3d^2}{5s^2}B - \frac{5d^2}{7s^2}C \,\&c.$$

$$L:b = L:a + \frac{nd}{s} + \frac{d^2}{3s^2}A + \frac{3d^2}{5s^2}B + \frac{5d^2}{7s^2}C \,\&c.$$

Where A, B, C, *&c.* denote the Term going before respectively. But we may give an Approximation sufficient for making the Logarithms to more Places than in most Tables now used. Thus

$$\left.\begin{array}{l} L:a = L:b - \dfrac{n \times 3ds}{3s^2 - d^2} \\ L:b = L:a + \dfrac{n \times 3ds}{3s^2 - d^2} \end{array}\right\} \text{Or,}$$

$$\left.\begin{array}{l} L:b - \dfrac{nd}{s} \times \dfrac{15s^2 - 4d^2}{15s^2 - 9d^2} \\ L:a + \dfrac{nd}{s} \times \dfrac{15s^2 - 4d^2}{15s^2 - 9d^2} \end{array}\right\} \textit{nearly.}$$

These Approximations are but *Corollaries* to a general Method of converting a Series converging ever so slow into another converging faster. We shall illustrate both by an *Example*, let the Logarithm of 11 be required, the Log. of 10 is 1, hence $s = 21$, $d = 1$; therefore by the Series this

OPERATION.

$$L:a = 1{,}00000000$$

$$\frac{nd}{s} = A = 0{,}04136138$$

$$\frac{d^2}{3s^2}A = B = \qquad 3126$$

$$\frac{3d^2}{5s^2}B = C = \qquad 5$$

$$L:a + A + B + C = 1{,}04139269 = \text{Log. } 11$$

T Or,

Or, By the Approximation $\frac{15s^2-4d^2}{15s^2-9d^2}=\frac{6611}{6606}$, and $\frac{d}{s}=\frac{1}{21}$, therefore $\frac{,8685889 6\times6611}{6606\times21}=,04139269$, to which add 1, the Log. of 10, then we have 1,04139269 for the Log. of 11, as found by the Series.

But if a and $a+2$ are two even Numbers, then $a+1$ is an odd Number, put $y=2a^2+4a+1$, then the Log. of $a+1$ is expressed by this Series, *viz.*

$$\text{L}:a+1=\text{L}:\frac{a\times\overline{a+2}}{2}+\frac{n}{2}\times\overline{\frac{1}{y}+\frac{1}{3y^3}+\frac{1}{5y^5}+\frac{1}{7y^7}},$$

&c. which converges so fast, that 'tis scarce worth while to use any Arts to render it more easy in practice; however, some may be desirous to have an Approximation, which is as follows, *viz.*

$$\text{L}:\overline{a+1}=\text{L}:\frac{a\times\overline{a+2}}{2}+n\times\frac{3y}{6y^2-2}\ \textit{nearly}; \text{ or (nearer}$$

$$\text{L}:\overline{a+1}=\text{L}:\frac{a\times\overline{a+2}}{2}+\frac{n}{6y^2}\times\frac{15y^2-4}{5y-3}$$

A TABLE *of the* Areas *of* Segments.

V. S.	*Seg. Area.*	*V. S.*	*Seg. Area.*
001	000042	034	008273
2	000119	35	008638
3	000219	36	009008
4	000337	37	009383
5	000470	38	009763
6	000618	39	010148
7	000779	40	010537
8	000951	41	010931
9	001135	42	011330
010	001329	43	011734
11	001533	44	012142
12	001746	45	012554
13	001968	46	012971
14	002199	47	013392
15	002438	48	013818
16	002685	49	014247
17	002940	50	014681
18	003202	51	015119
19	003471	52	015561
20	003748	53	016007
21	004031	54	016457
22	004322	55	016911
23	004618	56	017369
24	004921	57	017831
25	005230	58	018296
26	005546	59	018766
27	005867	60	019239
28	006194	61	019716
29	006527	62	020196
30	006865	63	020689
31	007209	64	021168
32	007558	65	021659
33	007913	66	022154

V. S.	*Seg. Area*	*V. S.*	*Seg. Area.*
067	022652	103	042687
68	023154	104	043296
69	023659	105	043908
70	024168	106	044522
71	024680	107	045139
72	025195	108	045759
73	025714	109	046381
74	026236	110	047005
75	026761	111	047632
76	027289	112	048262
77	027821	113	048894
78	028356	114	049528
79	028894	115	050165
80	029435	116	050804
81	029979	117	051446
82	030526	118	052092
83	031076	119	052736
84	031629	120	053385
85	032186	121	054036
86	032745	122	054689
87	033307	123	055345
88	033872	124	056003
89	034441	125	056663
90	035011	126	057326
91	035585	127	057991
92	036162	128	058658
93	036741	129	059327
94	037323	130	059999
95	038909	131	060672
96	038446	132	061348
97	039087	133	062026
98	039680	134	062707
99	040276	135	063389
100	040875	136	064074
101	041476	137	064760
102	042080	138	065449

V.S.	Seg. Area.	V.S.	Seg. Area.
139	066140	175	092313
140	066833	176	093074
141	067528	177	093836
142	068225	178	094601
143	068924	179	095366
144	069625	180	096134
145	070328	181	096903
146	071033	182	097674
147	071741	183	098447
148	072450	184	099221
149	073161	185	099990
150	073874	186	100774
151	074589	187	101553
152	075306	188	102334
153	076026	189	103116
154	076749	190	103900
155	077469	191	104685
156	078194	192	105472
157	078921	193	106261
158	079649	194	107051
159	080380	195	107842
160	081112	196	108636
161	081846	197	109430
162	082524	198	110226
163	083320	199	111024
164	084059	200	111823
165	084801	201	112624
166	085544	202	113426
167	086289	203	114230
168	087036	204	115035
169	087785	205	115842
170	088535	206	116650
171	089287	207	117460
172	090041	208	118271
173	090797	209	119000
174	091554	210	119897

V. S.	*Seg. Area.*	*V. S.*	*Seg. Area.*
211	120712	247	150953
212	121529	248	151816
213	122347	249	152680
214	123167	250	153546
215	123988	251	154412
216	124810	252	155280
217	125634	253	156149
218	126459	254	157019
219	127285	255	157890
220	128113	256	158762
221	128942	257	159636
222	129773	258	160510
223	130605	259	161386
224	131438	260	162263
225	132272	261	163140
226	133108	262	164019
227	133945	263	164899
228	134784	264	165780
229	135624	265	166663
230	136465	266	167546
231	137307	267	168430
232	138150	268	169315
233	138995	269	170202
234	139841	270	171089
235	140688	271	171978
236	141537	272	172867
237	142387	273	173758
238	143238	274	174649
239	144099	275	175542
240	144944	276	176435
241	145799	277	177330
242	146655	278	178225
243	147512	279	179122
244	148371	280	180019
245	149230	281	180918
246	150091	282	181817

V. S.	Seg. Area.	V. S.	Seg. Area.
283	182718	319	215733
284	183619	320	216666
285	184521	321	217599
286	185425	322	218533
287	186329	323	219468
288	187234	324	220404
289	188140	325	221401
290	189047	326	222277
291	189955	327	223215
292	190864	328	224154
293	191775	329	225093
294	192684	330	226033
295	193596	331	226974
296	194509	332	227915
297	195422	333	228858
298	196337	334	229801
299	197252	335	230644
300	198168	336	231389
301	199085	337	232034
302	200921	338	232579
303	201232	339	233026
304	201841	340	235473
305	202761	341	236421
306	203681	342	237369
307	204605	343	238318
308	205527	344	239268
309	206451	345	240218
310	207376	346	241169
311	208301	347	242121
312	209227	348	243074
313	210154	349	244026
314	211082	350	244980
315	212011	351	245934
316	212940	352	246889
317	213871	353	247845
318	214802	354	248801

V. S.	*Seg. Area.*	*V. S.*	*Seg. Area.*
355	249757	391	284568
356	250715	392	285544
357	251673	393	286521
358	252631	394	287498
359	253590	395	288476
360	254550	396	289453
361	255510	397	290432
362	256471	398	291411
363	257433	399	292390
364	258395	400	293369
365	259357	401	294349
366	260320	402	295330
367	261284	403	296311
368	262248	404	297292
369	263213	405	298273
370	264178	406	299255
371	265144	407	300238
372	266111	408	301220
373	267078	409	302203
374	268045	410	303187
375	269013	411	304171
376	269982	412	305155
377	270951	413	306138
378	271920	414	307125
379	272890	415	308110
380	273861	416	309095
381	274832	417	310081
382	275803	418	311068
383	276775	419	312054
384	277748	420	313041
385	278721	421	314029
386	279694	422	315016
387	280668	423	316004
388	281642	424	316992
389	282617	425	317981
390	283592	426	318970

S.V.	Seg. Area.	V.S.	Seg. Area.
427	319959	464	356730
428	320948	465	357727
429	321938	466	358725
430	322928	467	359723
431	323918	468	360721
432	324909	469	361719
433	325900	470	362717
434	326892	471	363715
435	327884	472	364713
436	328874	473	365712
437	329866	474	366710
438	330858	475	367709
439	331850	476	368708
440	332843	477	369707
441	333836	478	370706
442	334829	479	371705
443	335822	480	372704
444	336816	481	373703
445	337810	482	374702
446	338804	483	375702
447	339798	484	376702
448	340793	485	377701
449	341787	486	378701
450	342782	487	379700
451	343777	488	380700
452	344772	489	381699
453	345768	490	382699
454	346764	491	383699
455	347759	492	384699
456	348755	493	385699
457	349752	494	386699
458	350748	495	387699
459	351745	496	388699
460	352751	497	389699
461	353768	498	390699
462	354795	499	391699
463	355732	500	392699

It

It will be easy from what was said above, for the Reader to apply these Theorems of himself.

3dly. Seeing Silver, Soap, Candles, Starch, and several other Commodities, are charged according to Weight, and some Frauds having been committed by using false Scales, the Officer ought to take particular Care in that Point, as to see if the Beam play equally free on both Sides the Fulcrum; he ought also to see if the Brachia are equal in Length, and the Beam equally heavy on both Sides; for any of these being different, will cause the Weight to vary from Truth; for which Purpose 'tis thought convenient to add the following

PROPOSITION.

The Weights of any Body taken in each Scale of a Ballance of unequal Brachia's being given, to find its true Weight.

SOLUTION.

Multiply the two Weights together, and extract the Square Root of the Product, that will be the Weight of the Body proposed. *Fig.* 73.

For let the Weight of the Body be x, and when it is put in the Scale A, let it weigh w; but when put in the Scale B, let it weigh W; then from the known Law in *Mechan.* we have these Proportions, *viz.* DC : CE :: $w : x$, and DC : CE :: x : W, therefore $w : x :: x$: W, or $x^2 = w \times W$, that is $x = \sqrt{w \times W}$. *Q. E. O.*

Corol. Hence we may also find the Proportion of the Brachia's of these Scales, for DC : CE :: w : $\sqrt{w \times w}$.

N. B.

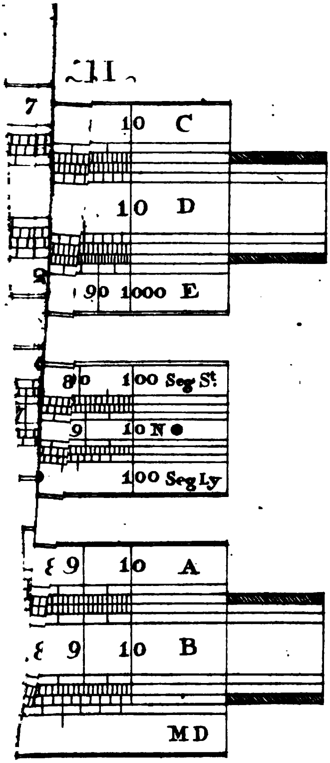
N. B. *'Tis thought proper to add the following Remark to* Scholium 2. Pag. 256.

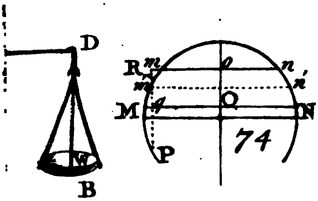
The Bottoms of Common Brewers large Backs do generally warp after they have been a little time ufed, and become more and more uneven as they grow older, efpecially fuch as are not every where well and equally fupported; many of them are fo large and uneven, that 'tis hardly poffible to find a true Medium of the Depth of the Wort, without taking a very great Number of Depths, and fome are fo fituate, that it is difficult to take Depths in every Part where it may be neceffary, and therefore it is a common Practice to take one Depth at a Place in the Back (clofe to the Side) which may be always moft conveniently come at; then fee all the Wort let down into a Tun, in which it may be exactly gauged, and (if the Dip taken in the Back does not happen to be a mean one) mark on the Side of the Back fuch Addition or Abatement as will make the Gauge of the Wort in the Back equal to the Gauge thereof in the Tun, which is called fetting a Back by feeing the Wort come down; and this is frequently done, becaufe moft large Backs are continually fettling more one way than another. Befides,

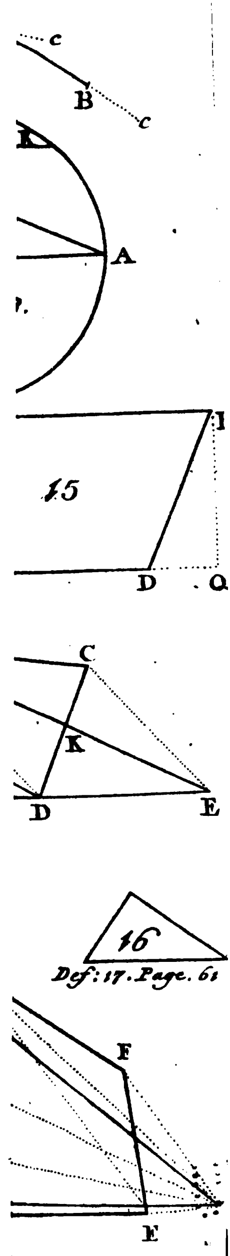
The Dimenfions of large Backs are (for want of more proper Inftruments) generally taken with inched Tapes, which are feldom very exactly divided, and the Alteration of Weather affects them in their Lengths; and a very fmall Error in the Dimenfions of fuch a large Veffel, caufes a confiderable one in its Area: but by fetting the Back as above, there will be a Compenfation in the Depth for any Error that may happen in taking the other Dimenfions. So we fhall here put a Period to this Treatife.

FINIS.

Plate. I







Will: Wyeth *Sculp.* 1740.

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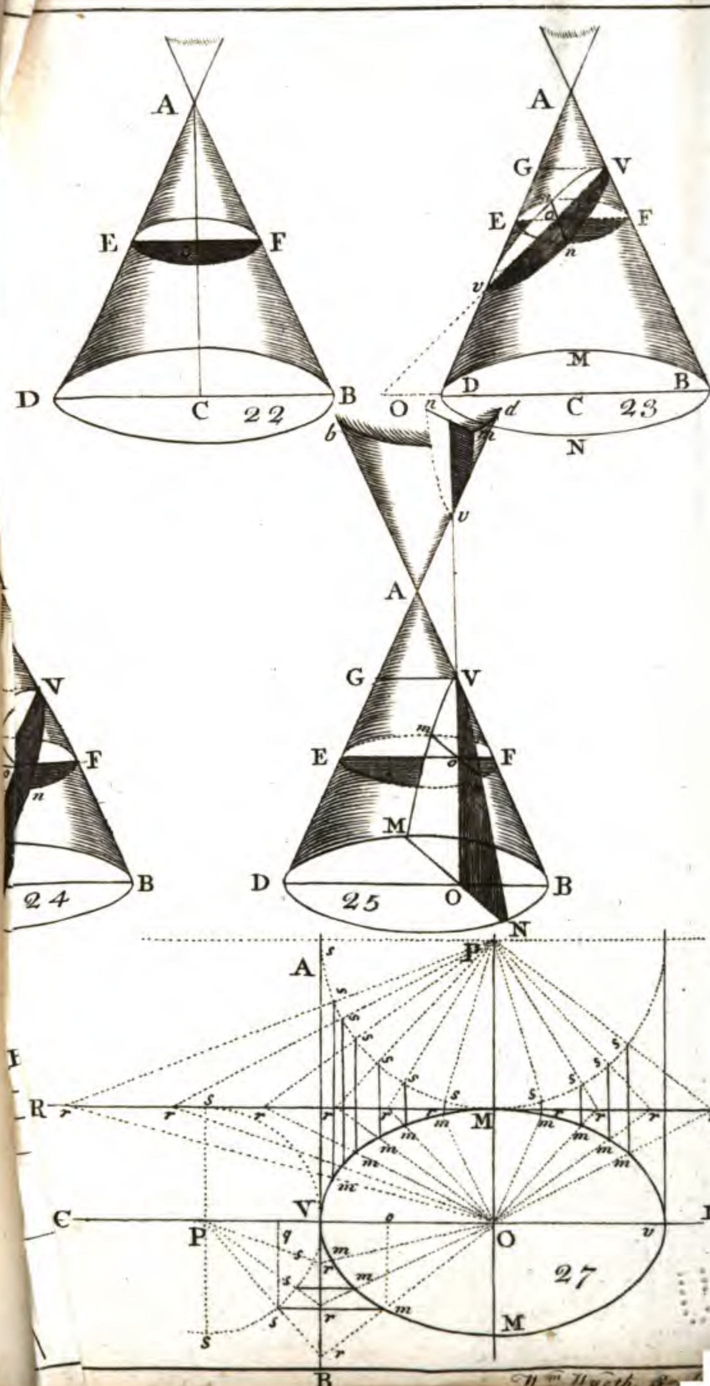
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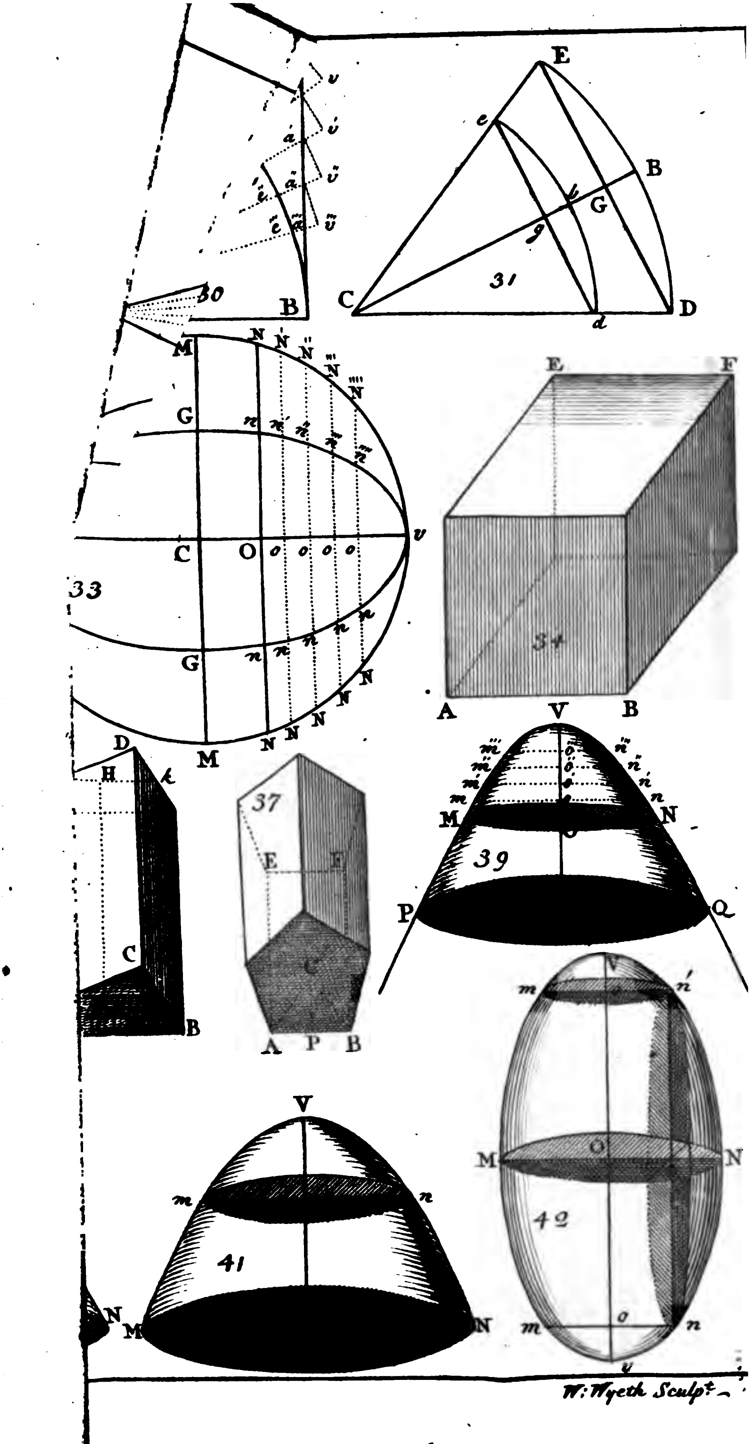
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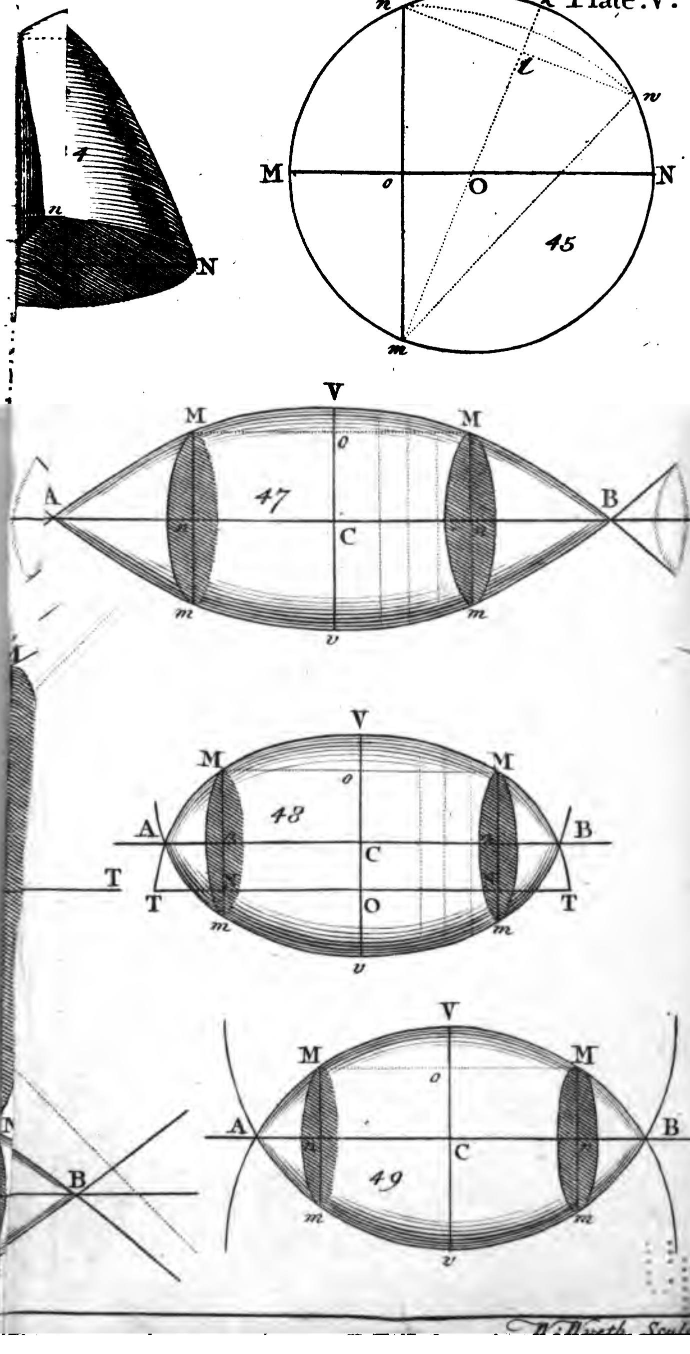
W: Wyeth Sculp^t.

Plate .V.



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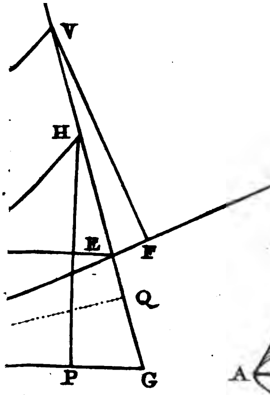
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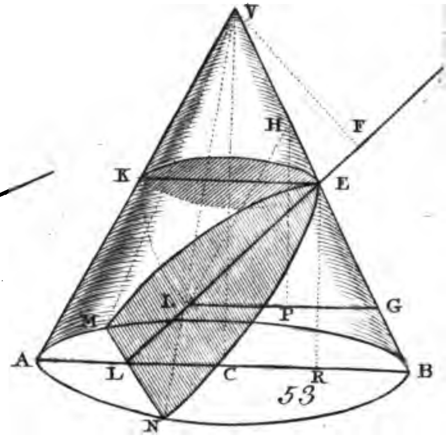
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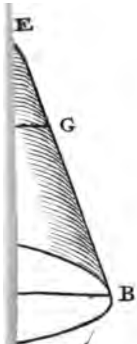
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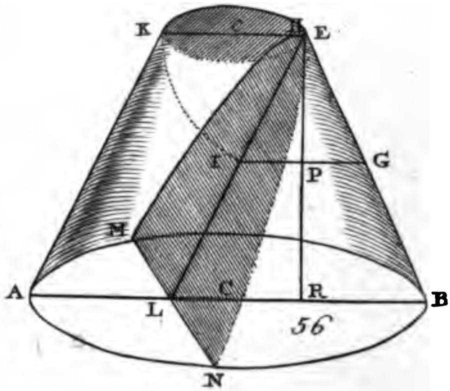
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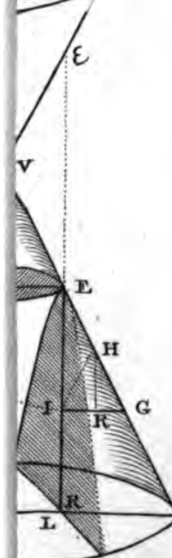
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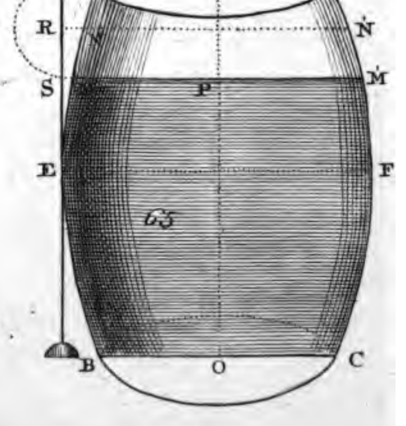
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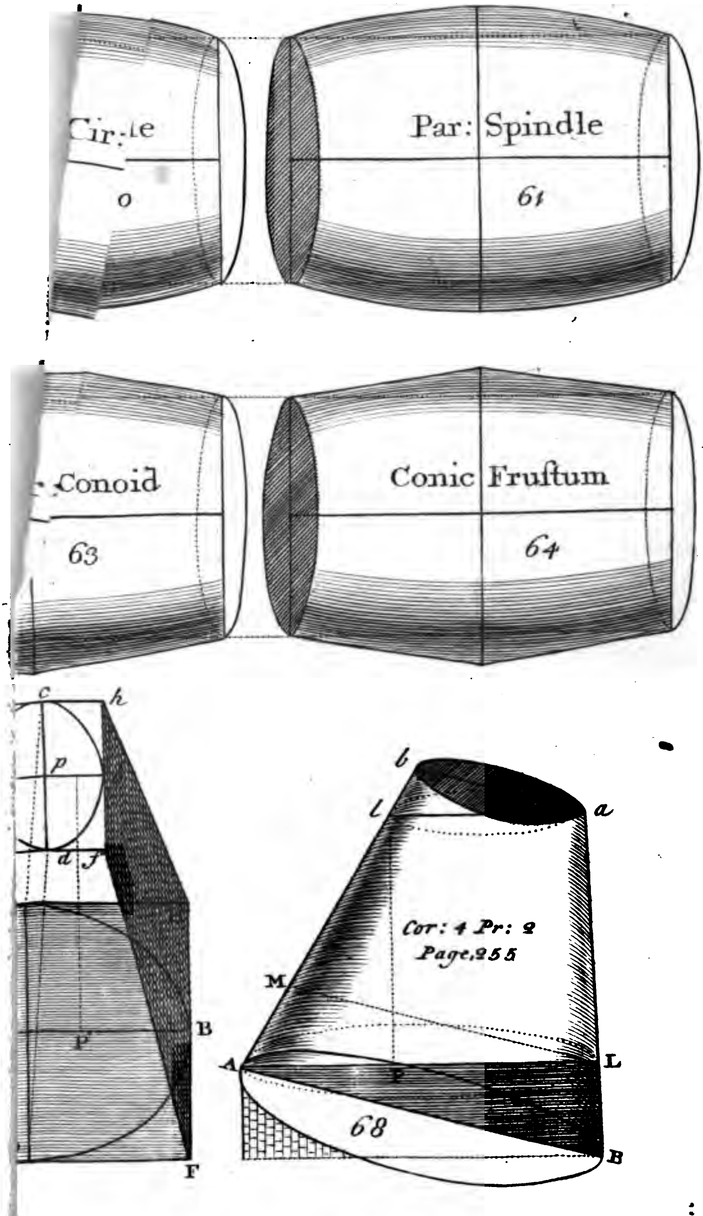
Plate. VII.



W: Wyeth Sculp.t